Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 7 (1996), 55–67.

DIFFERENTIAL AND INTEGRAL INEQUALITIES

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Dedicated to the memory of Professor Dragoslav S. Mitrinović

1. INTRODUCTION

Let $E : [0, +\infty) \to \mathbf{R}$ be a *nonnegative, non-increasing, locally absolutely continuous* function. Assume that there exists another *locally absolutely continuous* function $\rho : [0, +\infty) \to \mathbf{R}$ and there are three real numbers a, b and α such that

$$|\rho| \le aE \quad \text{in} \quad [0, +\infty) \tag{1}$$

and

$$\rho' \le -bE' - E^{\alpha+1} \quad \text{a.e. in} \quad [0, +\infty). \tag{2}$$

How can we estimate $E(t)$?

Problems of this type often appear during the study of dissipative linear evolutionary problems where E denotes the energy of the solution. It is sufficient to consider the case where $E(0) = 1$. Indeed, if $E(0) = 0$, then $E \equiv 0$. On the other hand, if $E(0) > 0$, then replacing E, ρ, a and b respectively by $E/E(0)$, $\rho E(0)^{-\alpha-1}$, $aE(0)^{-\alpha}$ and $aE(0)^{-\alpha}$, we obtain a solution of (1), (2) satisfying $E(0) = 1$. We will therefore assume in the sequel that

$$E(0) = 1. \tag{3}$$

Let us briefly recall the Liapunov method as usually applied to this problem (see e.g. [**1**], [**4**], [**5**], [**10**], [**11**]). Fix a real number d satisfying

$$d > a \quad \text{and} \quad d \ge b, \tag{4}$$

[0]1991 Mathematics Subject Classification: 26D10

and consider the function $F := dE + \rho$. One can readily verify that $F : [0, +\infty) \to \mathbf{R}$ is *nonnegative, non-increasing, locally absolutely continuous*. Furthermore,

$$0 \le (d-a)E \le F \le (d+a)E \quad \text{in} \quad [0, +\infty)$$

and

$$F' \le -(d+a)^{-\alpha-1} F^{\alpha+1} \quad \text{a.e. in} \quad [0, +\infty).$$

Dividing by $F^{\alpha+1}$ and integrating it follows that

$$F(t) \le \begin{cases} F(0)e^{-t/(d+a)} & \text{if } \alpha = 0; \\ (F(0)^{-\alpha} + \alpha(d+a)^{-\alpha-1}t)^{-1/\alpha} & \text{if } \alpha \ne 0 \end{cases}$$

and therefore

$$E(t) \le \begin{cases} \frac{d+a}{d-a} e^{-t/(d+a)} & \text{if } \alpha = 0; \\ \frac{d+a}{d-a} \left(\frac{d+a+\alpha t}{d+a} \right)^{-1/\alpha} & \text{if } \alpha \ne 0 \end{cases}$$

for all $t \ge 0$ such that $E(t) > 0$.

Next we minimize the right-hand side of this estimate with respect to d satisfying (4). Since (as we shall see at the end of this paper) this method does not lead to sharp estimates, we only consider henceforth the special case where

$$\alpha = 0 \qquad \text{and} \qquad a > 0.$$

Then we have

$$E(t) \le \frac{d+a}{d-a} e^{-t/(d+a)} =: f(d) \tag{5}$$

for all $t \ge 0$ and for all d satisfying (4). (Observe that this inequality makes sense and remains valid without the assumption $E(t) > 0$.)

An easy computation shows that

$$f'(d) = e^{-t/(d+a)} \frac{(t-2a)d - (t+2a)a}{(d-a)^2(d+a)}$$

Hence f is decreasing (resp. increasing) if $(t-2a)d - (t+2a)a < 0$ (resp. > 0).

If $0 \le t \le 2a$, then f is decreasing in $(a, +\infty)$ and tends to 1 as $t \to +\infty$. Therefore we only obtain the trivial estimate $E(t) \le 1$.

If $t > 2a$, then f decreases in (a, A) and increases in $(A, +\infty)$ where

$$A = \frac{t+2a}{t-2a} a \quad (> a).$$

We distinguish two cases:

If $b \le A$, then choosing $d = A$ in (5) we obtain that

$$E(t) \le \frac{t}{2a} e^{-(t-2a)/(2a)}.$$

If $b \geq A$, then choosing $d = b$ in (5) we conclude that

$$E(t) \leq \frac{b+a}{b-a} e^{-t/(b+a)}.$$

If $b \leq a$, then $b \leq A$ for all $t > 2a$. If $b > a$, then $b \leq A$ if and only if $2a < t \leq 2a\frac{b+a}{b-a}$.

We have thus proven the following:

Proposition 1. *If E, ρ solve $(1) - (3)$ with $\alpha = 0$ and $a > 0$, then we have the following estimates*:

$$E(t) \leq \begin{cases} 1 & \text{if } 0 \leq t \leq 2a; \\ \frac{t}{2a} e^{(2a-t)/(2a)} & \text{if } b \leq a \text{ and } t \geq 2a; \\ \frac{t}{2a} e^{(2a-t)/(2a)} & \text{if } b > a \text{ and } 2a \leq t \leq 2a\frac{b+a}{b-a}; \\ \frac{b+a}{b-a} e^{-t/(b+a)} & \text{if } b > a \text{ and } t \geq 2a\frac{b+a}{b-a}. \end{cases} \tag{6}$$

Despite the very frequent application of this method, the above estimates are not optimal. Applying a different method we shall prove

Theorem 2. a) *The problem $(1) - (3)$ has no solution unless $\alpha > -1$, $a \geq 0$ and $a + b > 0$.*

b) *If E, ρ solve (1)–(3) with some $\alpha > 0$, then we have the following estimates*:

b1) *If $-a < b \leq a$, then*

$$E(t) \leq \begin{cases} 1 & \text{if } 0 \leq t \leq (a+b); \\ \left(\frac{a+b+\alpha t}{(a+b)(1+\alpha)}\right)^{-1/\alpha} & \text{if } t \geq (a+b), \end{cases} \tag{7}$$

and in the second case the inequality is strict;

b2) *If $b > a$, then*

$$E(t) \leq \begin{cases} 1 & \text{if } 0 \leq t \leq 2a; \\ \left(\frac{a+b+\alpha t}{a+b+2\alpha a}\right)^{-1/\alpha} & \text{if } t \geq 2a. \end{cases} \tag{8}$$

c) *If E, ρ solve $(1) - (3)$ with $\alpha = 0$, then we have the following estimates*:

c1) *If $-a < b \leq a$, then*

$$E(t) \leq \begin{cases} 1 & \text{if } 0 \leq t \leq a+b; \\ e^{(a+b-t)/(a+b)} & \text{if } t \geq a+b, \end{cases} \tag{9}$$

and in the second case the inequality is strict;

c2) *If $b > a$, then*

$$E(t) \leq \begin{cases} 1 & \text{if } 0 \leq t \leq 2a; \\ e^{(2a-t)/(a+b)} & \text{if } t \geq 2a. \end{cases} \tag{10}$$

d) *If E, ρ solve* (1) – (3) *with some* $-1 < \alpha < 0$, *then we have the following estimates*:

d1) *If* $-a < b \le a$, *then*

$$E(t) \le \begin{cases} 1 & \text{if } 0 \le t \le (a+b); \\ \left(\frac{a+b+\alpha t}{(a+b)(1+\alpha)}\right)^{-1/\alpha} & \text{if } (a+b) \le t < (a+b)/|\alpha|; \\ 0 & \text{if } t \ge (a+b)/|\alpha|, \end{cases} \tag{11}$$

and in the second case the inequality is strict;

d2) *If* $b > a$, *then*

$$E(t) \le \begin{cases} 1 & \text{if } 0 \le t \le 2a; \\ \left(\frac{a+b+\alpha t}{a+b+2\alpha a}\right)^{-1/\alpha} & \text{if } 2a \le t \le (a+b)/|\alpha|; \\ 0 & \text{if } t \ge (a+b)/|\alpha|. \end{cases} \tag{12}$$

The above estimates are optimal.

Remark. Letting $\alpha \to 0$ in the formulae corresponding to $\alpha \ne 0$ we find the formulae for $\alpha = 0$.

For the proof of Theorem 2, we will have to study a closely related *integral inequality*, already used in [**2**], [**3**], [**6**]–[**9**]:

$$\int_t^{+\infty} E(s)^{\alpha+1}\, \mathrm{d}s \le TE(t), \qquad t \ge 0. \tag{13}$$

Here we only assume that $E : [0, +\infty) \to \mathbf{R}$ is a *nonnegative, non-increasing* (hence measurable) function and that α, T are given real numbers. If $E(0) = 0$, then $E \equiv 0$. If $E(0) > 0$, then replacing E by $E/E(0)$ and T by $TE(0)^{-\alpha}$ we obtain a solution of (13) such that $E(0) = 1$.

Furthermore, in order to avoid the trivial solution

$$E(t) = \begin{cases} 1 & \text{if } t = 0; \\ 0 & \text{if } t > 0, \end{cases}$$

we shall only consider consider solutions of (13) such that

$$E(0) = 1 \qquad \text{and} \qquad E \not\equiv 0 \;\text{ in }\; (0, \infty). \tag{14}$$

The following result, interesting in itself, completes some earlier theorems of Haraux [**2**], [**3**]:

Theorem 3. a) *The problem* (13) – (14) *has no solution unless* $\alpha > -1$ *and* $T > 0$.

b) *If E solves* (13) – (14) *with some* $\alpha > 0$, *then we have the following estimates*:

$$E(t) \le \begin{cases} 1 & \text{if } 0 \le t \le T; \\ \left(\frac{T+\alpha t}{T+\alpha T}\right)^{-1/\alpha} & \text{if } t \ge T. \end{cases} \tag{15}$$

Moreover, the second inequality is strict if E is right continuous.

d) *If E solves* $(13)-(14)$ *with $\alpha = 0$, then we have the following estimates:*

$$E(t) \le \begin{cases} 1 & \text{if } 0 \le t \le T; \\ e^{(T-t)/T} & \text{if } t \ge T. \end{cases} \tag{16}$$

Moreover, the second inequality is strict if E is right continuous.

e) *If E solves* $(13)-(14)$ *with some $-1 < \alpha < 0$, then we have the following estimates:*

$$E(t) \le \begin{cases} 1 & \text{if } 0 \le t \le T; \\ \left(\frac{T+\alpha t}{T+\alpha T}\right)^{-1/\alpha} & \text{if } T \le t < T/|\alpha|; \\ 0 & \text{if } t \ge T/|\alpha|. \end{cases} \tag{17}$$

Moreover, the second inequality is strict if E is right continuous.

These estimates are optimal.

Remark. As in the preceding results, letting $\alpha \to 0$ in the formulae corresponding to $\alpha \ne 0$we find the formulae for $\alpha = 0$.

2. PROOF Of THEOREM 3

If $\alpha \le -1$, then (13) is meaningful only if $E(t) > 0$ for all $t > 0$. However, then $E(s)^{\alpha+1} \ge E(0)^{\alpha+1} = 1$ for all $s \ge 0$ and therefore the integral on the left-hand side of (14) is infinite.

If $T \le 0$, then (13) implies at once that E vanishes in $(0, +\infty)$, contradicting (14).

Thus part a) of the theorem is proven. Henceforth we may therefore assume that $\alpha > -1$ and $T > 0$.

If $0 \le t \le T$, then the estimates $E(t) \le 1$ of (15)–(17) follow simply from the non-increasingness of E. Also, there is nothing to prove if $t \ge B$ where

$$B = \sup\{r \ge 0 \mid E(r) > 0\}.$$

We may thus assume that $T < t < B$.

The formula

$$F(r) = \int_r^{+\infty} E(s)^{\alpha+1} \, \mathrm{d}s$$

defines a nonnegative, non-increasing and locally absolutely continuous function $F : [0, \infty) \to \mathbf{R}$. It follows from (13) that

$$-F' \ge T^{-\alpha-1} F^{\alpha+1}$$

almost everywhere in $(0,\infty)$. Dividing by $F^{\alpha+1}$ and integrating in $(0,s)$, we obtain for every $0<s<B$ the following inequalities:

$$F(s)\le\begin{cases}(F(0)^{-\alpha}+\alpha T^{-\alpha-1}s)^{-1/\alpha} & \text{if } \alpha\ne 0;\\ F(0)e^{-s/T} & \text{if } \alpha=0.\end{cases}$$

Since $F(0)\le T$ by $(13)-(14)$, these inequalities remain valid if we replace $F(0)$ by T. Furthermore, we have

$$F(s)\ge\int_s^{T+(\alpha+1)s}E(r)^{\alpha+1}\,\mathrm{d}r\ge(T+\alpha s)E(T+(\alpha+1)s)^{\alpha+1}.$$

Therefore, we deduce from the preceding inequalities the estimates

$$(T+\alpha s)E(T+(\alpha+1)s)^{\alpha+1}\le\begin{cases}(T^{-\alpha}+\alpha T^{-\alpha-1}s)^{-1/\alpha} & \text{if } \alpha\ne 0;\\ Te^{-s/T} & \text{if } \alpha=0,\end{cases}$$

or equivalently,

$$E(T+(\alpha+1)s)\le\begin{cases}\left(\frac{T+\alpha s}{T}\right)^{-1/\alpha} & \text{if } \alpha\ne 0;\\ e^{-s/T} & \text{if } \alpha=0,\end{cases}$$

for all $0<s<B$.

If $\alpha\ge 0$, then these estimates obviously remain valid for all $s>0$. Choosing $s=\frac{t-T}{\alpha+1}$ hence $(15)-(16)$ follow.

If $-1<\alpha<0$, then the right-hand side of the above estimate is meaningless for $s\ge T/|\alpha|$. Hence $E(t)=0$ for all $t\ge T/|\alpha|$, proving the third inequality in (17). Furthermore, the above estimate obviously remains valid for all $0<s<T/|\alpha|$. Since $T<t<B$ implies that $0<\frac{t-T}{\alpha+1}<T/|\alpha|$, we may choose $s=\frac{t-T}{\alpha+1}$ in the above estimate, and the second inequality of (17) follows.

Now assume that E is right continuous and prove that the second inequalities of $(15)-(17)$ are strict. Assume on the contrary that we have equality in the second inequality of one of the formulae $(15)-(17)$ for some $t'\ge T$:

$$E(t')=\begin{cases}\left(\frac{T+\alpha t'}{T+\alpha T}\right)^{-1/\alpha} & \text{if } \alpha\ne 0;\\ e^{(T-t')/T} & \text{if } \alpha=0.\end{cases}\tag{18}$$

Using the right continuity of E in t', there is a constant $0<\beta<1$ such that

$$\int_0^{t'}E^{\alpha+1}\,\mathrm{d}s\le\beta\int_0^{+\infty}E^{\alpha+1}\,\mathrm{d}s.$$

It follows that the function

$$G(t)=\begin{cases}E(t) & \text{if } 0\le t\le t';\\ 0 & \text{if } t>t'\end{cases}$$

also satisfies (13) – (14), even if we replace the constant T in (13) by βT. Applying the already proved (weak) estimates (15) – (17), we have

$$G(t') \le \begin{cases} \left(\frac{\beta T+\alpha t'}{\beta T+\alpha\beta T}\right)^{-1/\alpha} & \text{if } \alpha \ne 0; \\ e^{(\beta T-t')/(\beta T)} & \text{if } \alpha = 0. \end{cases}$$

(Note that the third case in (17) cannot occur because $G(t') > 0$ by assumption.) Using (18) and the equality $G(t') = E(t') > 0$, it follows that

$$\begin{cases} \left(\frac{T+\alpha t'}{T+\alpha T}\right)^{-1/\alpha} \le \left(\frac{\beta T+\alpha t'}{\beta T+\alpha\beta T}\right)^{-1/\alpha} & \text{if } \alpha \ne 0; \\ e^{(T-t')/T} \le e^{(\beta T-t')/(\beta T)} & \text{if } \alpha = 0. \end{cases}$$

But both inequalities contradict the property $\beta < 1$.

Let us now turn to the proof of the optimality of the estimates (15) – (17). Fix $\alpha > -1$, $T > 0$ and $t' \ge 0$ arbitrarily. If $0 \le t' < T$, then we have to construct a solution of (13) – (14) such that $E(0) = E(t') = 1$. Choose simply

$$E(t) = \begin{cases} 1 & \text{if } 0 \le t \le T; \\ 0 & \text{if } t > T. \end{cases}$$

The verification of (13) is immediate: the case $t > T$ is trivial, while for $0 \le t \le T$ we have

$$\int_t^{+\infty} E(s)^{\alpha+1}\,\mathrm{d}s \le \int_t^T 1\,\mathrm{d}s \le T = TE(t).$$

We may even construct continuous examples, e.g.,

$$E(t) = \begin{cases} 1 & \text{if } 0 \le t \le t'; \\ (T-t)/(T-t') & \text{if } t' \le t \le T; \\ 0 & \text{if } t > T. \end{cases}$$

If $t' \ge T$ (for $\alpha \ge 0$) or $T \le t' < T/|\alpha|$ (for $-1 < \alpha < 0$), then we have to construct a solution of (13) – (14) such that

$$E(t') = \begin{cases} \left(\frac{T+\alpha t'}{T+\alpha T}\right)^{-1/\alpha} & \text{if } \alpha \ne 0; \\ e^{(T-t')/T} & \text{if } \alpha = 0. \end{cases}$$

If $\alpha = 0$, then let us choose

$$E(t) = \begin{cases} e^{-t/T} & \text{if } 0 \le t \le t'-T; \\ e^{-(t'-T)/T} & \text{if } t'-T \le t \le t'; \\ 0 & \text{if } t > t'. \end{cases}$$

If $\alpha \ne 0$, then let us choose

$$E(t) = \begin{cases} \left(\frac{T+\alpha t}{T}\right)^{-1/\alpha} & \text{if } 0 \le t \le \frac{t'-T}{\alpha+1}; \\ \left(\frac{T+\alpha t'}{T+\alpha T}\right)^{-1/\alpha} & \text{if } \frac{t'-T}{\alpha+1} \le t \le t'; \\ 0 & \text{if } t > t'. \end{cases}$$

(Note that these functions are not continuous.)

The only nontrivial property to verify is (13) for $0 \le t \le \frac{t'-T}{\alpha+1}$. Since $E^{\alpha+1} = -TE'$ in $(0, \frac{t'-T}{\alpha+1})$ in all cases, we have in fact equality:

$$\begin{aligned}\int_t^{+\infty} E(s)^{\alpha+1}\ \mathrm{d}s &= \int_t^{(t'-T)/(\alpha+1)} E(s)^{\alpha+1}\ \mathrm{d}s + \int_{(t'-T)/(\alpha+1)}^{t'} E(s)^{\alpha+1}\ \mathrm{d}s \\ &= TE(t) - TE\left(\frac{t'-T}{\alpha+1}\right) + \left(t' - \frac{t'-T}{\alpha+1}\right) E\left(\frac{t'-T}{\alpha+1}\right)^{\alpha+1} \\ &= TE(t).\end{aligned}$$

The proof of Theorem 3 is completed.

3. PROOF OF THEOREM 2

We begin with a lemma relating the problem (1)–(3) to the integral inequality (13) – (14).

Lemma 4. *If E, ρ solve (1) – (3) with some a, b and α, then E also solves (13) – (14) with the same α and with $T = a + b$.*

Proof. Since the solutions E of (1) – (3) are continuous, (3) implies (14).

It follows from (1) – (2) and from the non-increasingness of E that

$$\int_t^{t'} E(s)^{\alpha+1}\ \mathrm{d}s \le [bE+\rho]_{t'}^{t} \le 2(|a|+|b|)E(t) \tag{19}$$

for all $0 \le t < t' < +\infty$. Letting $t' \to +\infty$ hence we conclude that

$$\int_t^{+\infty} E(s)^{\alpha+1}\ \mathrm{d}s \le 2(|a|+|b|)E(t)$$

for all $t \ge 0$. Applying Theorem 3 it follows that $E(t') \to 0$ as $t' \to \infty$. Using (1) we also obtain that $\rho(t') \to 0$ as $t' \to +\infty$. Hence, letting $t' \to \infty$ in the first inequality of (19), we conclude that

$$\int_t^{+\infty} E(s)^{\alpha+1}\ \mathrm{d}s \le bE(t) + \rho(t).$$

Applying (1) again, hence (13) follows. □

It follows at once from (1) and (3) that $a \ge 0$. The rest of part a and parts $b1$, $c1$, $d1$ Theorem 2 follow at once from Lemma 4 and Theorem 3, including the strict inequalities.

It remains to prove the estimates (8), (10) and (12). Since the inequality $E(t) \le 1$ is obvious, we have to prove for $\alpha > -1$, $b > a \ge 0$ and $t > 2a$ the following estimates:

$$(20)\qquad E(t) \le \begin{cases} \left(\frac{a+b+\alpha t}{a+b+2\alpha a}\right)^{-1/\alpha} & \text{if } \alpha > 0; \\ e^{(2a-t)/(a+b)} & \text{if } \alpha = 0; \\ \left(\frac{a+b+\alpha t}{a+b+2\alpha a}\right)^{-1/\alpha} & \text{if } \alpha < 0 \text{ and } t \le (a+b)/|\alpha|; \\ 0 & \text{if } \alpha < 0 \text{ and } t > (a+b)/|\alpha|. \end{cases}$$

Clearly, we may also assume that

$$t < B := \sup\{r \ge 0 \mid E(r) > 0\}.$$

Dividing the inequality (2) by $E^{\alpha+1}$, then integrating in $(0,t)$ and using (1), we obtain that

$$\begin{aligned} \int_0^t bE^{-\alpha-1}E' \, \mathrm{d}s \le & \int_0^t -1 - \rho' E^{-\alpha-1} \, \mathrm{d}s \\ &= [-\rho E^{-\alpha-1}]_0^t + \int_0^t -1 - (\alpha+1)\rho E^{-\alpha-2}E' \, \mathrm{d}s \\ &\le aE(t)^{-\alpha} + aE(0)^{-\alpha} - t - (\alpha+1)a\int_0^t E^{-\alpha-1}E' \, \mathrm{d}s, \end{aligned}$$

whence

$$(b+a+\alpha a)\int_0^t E^{-\alpha-1}E' \, \mathrm{d}s \le aE(t)^{-\alpha} + aE(0)^{-\alpha} - t.$$

Computing the integral, it follows easily that

$$E(t) \le \begin{cases} \left(\frac{a+b+\alpha t}{a+b+2\alpha a}\right)^{-1/\alpha} & \text{if } \alpha > 0; \\ e^{(2a-t)/(a+b)} & \text{if } \alpha = 0; \\ \left(\frac{a+b+\alpha t}{a+b+2\alpha a}\right)^{-1/\alpha} & \text{if } \alpha < 0. \end{cases}$$

Comparing with (20), it only remains to show that $E(t) = 0$ if $\alpha < 0$ and $t > (a+b)/|\alpha|$. Let us observe that for $\alpha < 0$ the right-hand side of the last inequality vanishes for $t = (a+b)/|\alpha|$. It cannot occur if $E(t) > 0$, therefore $E((a+b)/|\alpha|) = 0$ and our claim follows.

Now we are going to prove the optimality of our estimates (7) – (12). Fix $\alpha > -1$, $a \ge 0$, $b > -a$ arbitrarily. Furthermore, fix $t' \ge 0$ arbitrarily if $\alpha \ge 0$ and fix $0 \le t' < (a+b)/|\alpha|$ arbitrarily if $-1 < \alpha < 0$.

Let us define a number R in the following way: set

$$R = \begin{cases} 0 & \text{if } b \le a \text{ and } t' < a+b; \\ 0 & \text{if } b > a \text{ and } t' < 2a; \\ \frac{(a+b)(t'-2a)}{a+b+2\alpha a} & \text{if } b > a \text{ and } t' \ge 2a. \end{cases}$$

Furthermore, choose an arbitrary number

$$\frac{t'-a-b}{1+\alpha} < R \le t' \tag{21}$$

if $b \le a$ and $t' \ge a+b$; its value will be precised later.

These definitions are correct and $0 \le R \le t'$ in all cases.

Next we define the function E. For $\alpha > 0$ we set

$$E(t) = \begin{cases} \left(\frac{a+b+\alpha t}{a+b}\right)^{-1/\alpha} & \text{if } 0 \le t \le R; \\ E(R) & \text{if } R < t \le t'; \\ E(R)\left(1 + \frac{\alpha(t-t')E(R)^\alpha}{a+b-(t'-R)E(R)^\alpha}\right)^{-1/\alpha} & \text{if } t > t'. \end{cases}$$

For $\alpha = 0$ we define

$$E(t) = \begin{cases} e^{-t/(a+b)} & \text{if } 0 \le t \le R; \\ E(R) & \text{if } R < t \le t'; \\ E(R)e^{(t'-t)/(a+b+R-t')} & \text{if } t > t'. \end{cases}$$

Finally, for $-1 < \alpha < 0$ we set

$$E(t) = \begin{cases} \left(\frac{a+b+\alpha t}{a+b}\right)^{-1/\alpha} & \text{if } 0 \le t \le R; \\ E(R) & \text{if } R < t \le t'; \\ E(R)\left(1 + \frac{\alpha(t-t')E(R)^\alpha}{a+b-(t'-R)E(R)^\alpha}\right)^{-1/\alpha} & \text{if } t' < t < t''; \\ 0 & \text{if } t \ge t'', \end{cases}$$

where

$$t'' = t' + \frac{a+b-(t'-R)E(R)^\alpha}{|\alpha|E(R)^\alpha}.$$

If $0 \le t \le t'$, then $a+b+\alpha t > 0$; hence $E(t)$ is correctly defined and strictly positive. In particular, $E(R) > 0$. Let us show that

$$(t'-R)E(R)^\alpha < a+b \tag{22}$$

and

$$(t'-R)E(R)^\alpha \le 2a. \tag{23}$$

Indeed, if $b \le a$ and $t' < a+b$, then

$$(t'-R)E(R)^\alpha = t' < a+b \le 2a.$$

If $b > a$ and $t' < 2a$, then

$$(t'-R)E(R)^\alpha = t' < 2a < a+b.$$

If $b > a$ and $t' \geq 2a$, then

$$(t' - R)E(R)^\alpha = 2a < a + b$$

by a simple computation. Finally, if $b \leq a$ and $t' \geq a + b$, then

$$(t' - R)E(R)^\alpha = \frac{(t' - R)(a + b)}{a + b + \alpha R} < a + b \leq 2a$$

because $R > (t' - a - b)/(1 + \alpha)$ (see (21)).

Using (22) one can readily verify that E is a correctly defined, nonnegative, non-increasing, locally absolutely continuous function for all $t \geq 0$, and $E(0) = 1$. Let us assume for the moment the existence of a locally absolutely continuous function ρ satisfying (1)$-$(2), and prove the optimality of the estimates of Theorem 2.

Let us compute $E(t') = E(R)$. If $b > a$, then

$$E(t') = \begin{cases} 1 & \text{if } t' < 2a; \\ \left(\frac{a+b+\alpha t'}{a+b+2\alpha a}\right)^{-1/\alpha} & \text{if } t' \geq 2a \text{ and } \alpha \neq 0; \\ e^{(2a-t')/(a+b)} & \text{if } t' \geq 2a \text{ and } \alpha = 0. \end{cases}$$

This proves the optimality of the estimates (8), (10), (12). If $b \leq a$, then

$$E(t') = \begin{cases} 1 & \text{if } t' < a + b; \\ \left(\frac{a+b+\alpha R}{a+b}\right)^{-1/\alpha} & \text{if } t' \geq a + b \text{ and } \alpha \neq 0; \\ e^{-R/(a+b)} & \text{if } t' \geq a + b \text{ and } \alpha = 0. \end{cases}$$

Letting $R \to (t' - a - b)/(1 + \alpha)$ (see (21)) hence the optimality of the estimates (7), (9), (11) follows.

It remains to construct a locally absolutely continuous function $\rho : [0, +\infty) \to \mathbf{R}$ satisfying (1) and (2). Define

$$\rho(t) = \begin{cases} aE(t) & \text{if } 0 \leq t \leq R; \\ aE(R) - (t - R)E(R)^{\alpha+1} & \text{if } R \leq t \leq t'; \\ (a - (t' - R)E(R)^\alpha)E(t) & \text{if } t \geq t'. \end{cases}$$

Then ρ is locally absolutely continuous. The property (1) is obvious for $0 \leq t \leq R$; for $t > R$ it follows easily using (23):

$$aE(t) \geq \rho(t) \geq (a - (t' - R)E(R)^\alpha)E(t) \geq -aE(t).$$

Next we claim that

$$\rho' = -bE' - E^{\alpha+1} \quad \text{a.e. in} \quad [0, +\infty);$$

in particular, (2) is satisfied. Indeed, in $(0, R)$ we have

$$(bE' + \rho')(t) = (a + b)E'(t) = -E(t)^{\alpha+1}.$$

In (R, t') we have

$$(bE' + \rho')(t) = -E(R)^{\alpha+1} = -E(t)^{\alpha+1}.$$

In $(t', +\infty)$ we have

$$(bE' + \rho')(t) = (a + b - (t' - R)E(R)^{\alpha})E'(t) = -E(t)^{\alpha+1}$$

by another simple computation.

The proof of Theorem 2 is completed.

4. COMPARISON OF PROPOSITION 1 AND THEOREM 2

We are going to show that the estimates of Proposition 1 are optimal only in trivial cases. As in Proposition 1, assume that $\alpha = 0$ and $a > 0$.

a) If $b \le -a$, then (1) – (3) has no solution; this was not revealed by the Liapunov method: we only obtained in this case the estimate

$$E(t) \le \begin{cases} 1 & \text{if } 0 \le t \le 2a; \\ \frac{t}{2a} e^{(2a-t)/(2a)} & \text{if } t \ge 2a \end{cases}$$

(cf. (6)).

b) If $-a < b \le a$, then we have to compare the estimates (6) and (9). For $0 \le t \le a + b$ they both give $E(t) \le 1$. For $a + b < t \le 2a$ the estimate (9) is better because

$$e^{(a+b-t)/(a+b)} < 1.$$

Finally, for $t > 2a$ the estimate (9) is better again because

$$e^{(a+b-t)/(a+b)} < \frac{t}{2a} e^{(2a-t)/(2a)}.$$

Indeed, we have

$$e^{(a+b-t)/(a+b)} \le e^{(2a-t)/(2a)} < \frac{t}{2a} e^{(2a-t)/(2a)}.$$

c) If $b > a$, then we have to compare the estimates (6) and (10). For $0 \le t \le 2a$ they both give $E(t) \le 1$.

In order to show that for $t \ge 2a\frac{b+a}{b-a}$ the estimate (10) is better than (6), we have to prove that

$$e^{(2a-t)/(a+b)} < \frac{b+a}{b-a} e^{-t/(a+b)}. \tag{24}$$

Putting $x = 2a/(a + b)$ we have $0 < x < 1$, and the inequality takes the form $e^x < 1/(1 - x)$. This inequality is trivially satisfied:

$$e^x = \sum_{i=1}^{\infty} \frac{x^i}{i!} < \sum_{i=1}^{\infty} x^i = 1/(1 - x).$$

Finally, in order to show that for $2a < t \le 2a\frac{b+a}{b-a}$ the estimate (10) is better than (6), we have to prove the inequality

$$e^{(2a-t)/(a+b)} < \frac{t}{2a} e^{(2a-t)/(2a)}.$$

Keeping a and t fixed, let us increase b until $t = 2a\frac{b+a}{b-a}$ (then the left-hand side of the inequality increases). Then our inequality coincides with (24) and the claim follows.

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(Received September 15, 1995)