UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. 7 (1996), 36–44.

APPROXIMATION THEOREMS FOR SOME OPERATORS OF THE SZASZ–MIRAKJAN TYPE IN EXPONENTIAL WEIGHT SPACES

L. Rempulska, M. Skorupka

In this note we define some linear positive operators A_n and B_n of the Szasz–Mirakjan type in the space of continuous functions having exponential growth an infinity. In Sec. 2 we give some auxiliary results. In Sec. 3 we prove two approximations theorems for these operators.

1. PRELIMINARIES

1.1. Let $C \equiv C(R_0)$ be the set of all real–valued functions continuous on $R_0 := [0, +\infty)$. Analogously as in [**1**] for $p > 0$ we define

$$w_p(x) := e^{-px}, \qquad x \in R_0, \tag{1}$$

$$C_p := \{f \in C : \quad w_p \cdot f \text{ is uniformly continuous and bounded on } R_0\},$$

$$\|f\|_{C_p} := \sup_{x \in R_0} \; w_p(x)\,|f(x)|. \tag{2}$$

For $f \in C_p$, $p > 0$, and for $\delta > 0$ and $0 < \alpha \leq 1$ we define the modulus continuity $\omega(f, C_p; \delta)$ and the class Lip (C_p, α) ([**2**])

$$\omega(f, C_p; \delta) := \sup_{0<h\leq\delta} \|f(\cdot + h) - f(\cdot)\|_{C_p},$$

$$\text{Lip}(C_p, \alpha) := \Big\{ f \in C_p : \; \omega(f, C_p; \delta) = O(\delta^\alpha) \text{ as } \delta \to 0+ \Big\}.$$

It is easily observed that if $q > p > 0$, then $C_p \subset C_q$ and $\|f\|_{C_q} \leq \|f\|_{C_p}$ for every $f \in C_p$.

1.2. The SZASZ–MIRAKJAN operators

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{+\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \qquad x \in R_0, \quad n \in \mathbf{N},$$

[0]1991 Mathematics Subject Classification: 41A36

($\mathbf{N} := \{1, 2, \ldots\}$), for functions $f \in C_p$ in the norm of the space C_q, $q > p$, are examined in [**1**].

In our paper we introduce the linear positive operators A_n and B_n of the Szasz–Mirakjan type in the space C_p

$$A_n(f;x) := \frac{1}{1+\sinh nx}\left(f(0) + \sum_{k=0}^{+\infty} \frac{(nx)^{2k+1}}{(2k+1)!}\, f\left(\frac{2k+1}{n}\right)\right), \tag{3}$$

$$B_n(f;x) := \frac{1}{1+\sinh nx}\left(f(0) + \sum_{k=0}^{+\infty} \frac{(nx)^{2k+1}}{(2k+1)!}\, \frac{n}{2} \int\limits_{I_{n,k}} f(t)\,\mathrm{d}t\right), \tag{4}$$

$n \in \mathbf{N}$, $x \in R_0$, where $I_{n,k} := \left[\dfrac{2k+1}{n}, \dfrac{2k+3}{n}\right]$ and $\sinh x$, $\cosh x$ are the elementary hyperbolic functions, i.e. $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

The operators A_n and B_n are well–defined for all $f \in C_p$, $p > 0$. A_n and B_n are an operators from C_p into C_q for any $q > p$, provided n is large enough. In Sec. 2 we shall give some properties of these operators. In Sec. 3 we shall two direct approximation theorems for A_n and B_n using the modulus of continuity of function $f \in C_p$. These theorems are similar to suitable results given in [**1**] for the Szasz–Mirakjan operators S_n.

In Sec. 2 and 3 by $M_{p,q}$ we shall denote some suitable positive constants depending only on indicated parameters p, q.

2. AUXILIARY RESULTS

Denote by

$$S(nx) := \frac{\sinh nx}{1+\sinh nx}, \qquad T(nx) := \frac{\cosh nx}{1+\sinh nx}, \tag{5}$$

for $x \geq 0$ and $n \in \mathbf{N}$. By elementary calculations from (3)–(5) we obtain the following two lemmas.

Lemma 1. *For each $n \in \mathbf{N}$ and $x \in R_0$ we have*

$$\begin{aligned}
&A_n(1;x) = 1, \quad B_n(1;x) = 1, \quad A_n(t;x) = xT(nx),\\
&B_n(t;x) = A_n(t;x) + \frac{1}{n}\, S(nx), \quad A_n(t^2;x) = x^2 S(nx) + \frac{x}{n}\, T(nx),\\
&B_n(t^2;x) = A_n(t^2;x) + \frac{2}{n}\, A_n(t;x) + \frac{4}{3n^2}\, S(nx)\\
&\qquad = \left(x^2 + \frac{4}{3n^2}\right) S(nx) + \frac{3x}{n}\, T(nx).
\end{aligned} \tag{6}$$

Lemma 2. *For all* $n \in \mathbf{N}$, $x \in R_0$ *and* $p > 0$ *we have*

$$A_n(e^{pt};x) = \frac{1+\sinh\left(e^{p/n}\,nx\right)}{1+\sinh nx}, \quad A_n(te^{pt};x) = xe^{p/n}\,\frac{\cosh\left(e^{p/n}\,nx\right)}{1+\sinh nx},$$

$$A_n(t^2e^{pt};x) = x^2e^{p/n}\,\frac{\sinh\left(e^{p/n}\,nx\right)}{1+\sinh nx} + \frac{x}{n}\,e^{p/n}\,\frac{\cosh\left(e^{p/n}\,nx\right)}{1+\sinh nx},$$

$$B_n(e^{pt};x) = \frac{n}{2p}\left(e^{2p/n}-1\right)A_n(e^{pt};x) + \left(1-\frac{n}{2p}\left(e^{2p/n}-1\right)\right)\frac{1}{1+\sinh nx},$$

$$B_n(te^{pt};x) = \frac{n}{2p}\left(e^{2p/n}-1\right)A_n(te^{pt};x) + \frac{1}{p}\,e^{2p/n}A_n(e^{pt};x)$$
$$-\frac{1}{p}\,B_n(e^{pt};x) - \frac{1}{p}\left(e^{2p/n}-1\right)\frac{1}{1+\sinh nx},$$

$$B_n(t^2e^{pt};x) = \frac{n}{2p}\left(e^{2p/n}-1\right)A_n(t^2e^{pt};x) + \frac{2}{p}\,e^{2p/n}A_n(te^{pt};x)$$
$$+\frac{2}{np}\,A_n(e^{pt};x) - \frac{2}{p}\,B_n(te^{pt};x) - \frac{2}{np}\,\frac{1}{1+\sinh nx},$$

(7) $$A_n\left((t-x)^2e^{pt};x\right) = A_n(t^2e^{pt};x) - 2xA_n(te^{pt};x) + x^2A_n(e^{pt};x)$$
$$= 2e^{p/n}x^2\,\frac{\sinh\left(e^{p/n}nx\right)-\cosh\left(e^{p/n}nx\right)}{1+\sinh nx}$$
$$+x^2\,\frac{1+(1-e^{p/n})\sinh\left(e^{p/n}nx\right)}{1+\sinh nx} + \frac{x}{n}\,e^{p/n}\,\frac{\cosh\left(e^{p/n}nx\right)}{1+\sinh nx},$$

(8) $$B_n\left((t-x)^2e^{pt};x\right) = B_n(t^2e^{pt};x) - 2xB_n(te^{pt};x) + x^2B_n(e^{pt};x)$$
$$= \frac{n}{2p}\left(e^{2p/n}-1\right)A_n\left((t-x)^2e^{pt};x\right)$$

$$+\frac{2}{p}\left(e^{2p/n}-\frac{n}{2p}\left(e^{2p/n}-1\right)\right)A_n\left((t-x)e^{pt};x\right)$$

$$+\left(\frac{2}{np}-\frac{2}{p^2}\,e^{2p/n}-\frac{n}{p^3}\left(e^{2p/n}-1\right)\right)A_n\left(e^{pt};x\right)$$

$$+\left(1-\frac{n}{2p}\left(e^{2p/n}-1\right)\right)\frac{x^2}{1+\sinh nx}$$

$$+\frac{2}{p}\left(e^{2p/n}-2+\frac{n}{p}\left(e^{2p/n}-1\right)\right)\frac{x}{1+\sinh nx}$$

$$+\left(-\frac{2}{np}+\frac{2}{p^2}\left(e^{2p/n}-\frac{n}{2p}\left(e^{2p/n}-1\right)\right)\right)\frac{1}{1+\sinh nx}.$$

Using Lemmas 1 and 2, we shall prove some inequalities.

Lemma 3. *For all $x \in R_0$ and $n \in \mathbf{N}$ holds*

$$A_n\left((t-x)^2;x\right) \le 4\,\frac{x+1}{n}, \tag{9}$$

$$B_n\left((t-x)^2;x\right) \le \frac{31}{3}\,\frac{x+1}{n}. \tag{10}$$

Proof. By (3)–(5) and Lemma 1 for all $x \ge 0$ and $n \in \mathbf{N}$ we have

$$\begin{aligned} A_n\left((t-x)^2;x\right) &= A_n(t^2;x) - 2xA_n(t;x) + x^2A_n(1;x) \\ &= x^2\big(1+S(nx)-2T(nx)\big) + \frac{x}{n}\,T(nx) \end{aligned}$$

and analogously

$$B_n\left((t-x)^2;x\right) = A_n\left((t-x)^2;x\right) - \frac{2x}{n}\left(T(nx)-S(nx)\right) + \frac{4}{3n^2}\,S(nx).$$

Since $1-e^{-nx} \ge 0$ and $|1-2e^{-nx}| \le 1$ for $x \ge 0$ and $n \in \mathbf{N}$, we get

$$\frac{1}{1+\sinh nx} \le \frac{2}{e^{nx}+1} \le \frac{2}{e^{nx}}, \tag{11}$$

$$x^2|1+S(nx)-2T(nx)| = \frac{x^2\,|1-2e^{-nx}|}{1+\sinh nx} \le \frac{2x^2}{e^{nx}} \le \frac{4}{n^2},$$

$$0 < T(nx) \le 1, \qquad 0 \le S(nx) \le 1, \tag{12}$$

for $x \ge 0$ and $n \in \mathbf{N}$. From these we immediately obtain (9) and (10).

Lemma 4. *Suppose that $p > 0$, $q > p$ and $n_0 = n_0(p,q)$ be a fixed natural number such that*

$$n_0 > p\left(\ln\frac{q}{p}\right)^{-1}. \tag{13}$$

Then there exists a positive constant $M_{p,q}$ depending only on p,q such that

$$\left\|A_n(e^{pt};\cdot)\right\|_{C_q} \le 2, \tag{14}$$

$$\left\|B_n(e^{pt};\cdot)\right\|_{C_q} \le 2(p+1)e^{2p}, \tag{15}$$

$$w_q(x)\left|A_n\left((t-x)e^{pt};x\right)\right| \le M_{p,q}\,\frac{x+1}{n}, \tag{16}$$

$$w_q(x)A_n\left((t-x)^2e^{pt};x\right) \le M_{p,q}\,\frac{x+1}{n}\,, \tag{17}$$

$$w_q(x)B_n\left((t-x)^2e^{pt};x\right) \le M_{p,q}\,\frac{x+1}{n}\,, \tag{18}$$

for all $x \ge 0$ and $n \ge n_0$.

Proof. Let $p > 0$ and $q > p$ be a fixed numbers. Similarly as in [**1**] we write

$$p_n := n\left(e^{p/n}-1\right), \qquad n \in \mathbf{N}. \tag{19}$$

The sequence $(p_n)_1^\infty$ is decreasing and

$$p < p_n < pe^{p/n} \le pe^p \ \text{ for } \ n \in \mathbf{N}. \tag{20}$$

If n_0 is a fixed integer satisfying (13), then

$$q > pe^{p/n_0} > p_{n_0} > p_n \ \text{ for } \ n > n_0. \tag{21}$$

By (1), (11) and (19) we have

$$w_q(x)\,\frac{\sinh\left(e^{p/n}nx\right)}{1+\sinh nx} \le e^{-(q-p_n)x}, \tag{22}$$

$$w_q(x)\,\frac{\cosh\left(e^{p/n}nx\right)}{1+\sinh nx} \le 2e^{-(q-p_n)x}, \ \text{ for } \ x \ge 0,\ n \in \mathbf{N}, \tag{23}$$

which by Lemma 2 and (20)–(21) yields

$$w_q(x)A_n(e^{pt};x) = e^{-qx}\,\frac{1+\sinh\left(e^{p/n}nx\right)}{1+\sinh nx} \le 1+e^{-(q-p_n)x} \le 2 \ \text{ for } \ x \ge 0,\ n \ge n_0.$$

From this and (2) follows (14).

We observe that for $p > 0$ and $n \in \mathbf{N}$ holds

$$0 < e^{2p/n}-1 \le \frac{2p}{n}\,e^{2p/n}, \qquad \left|1-\frac{n}{2p}\left(e^{2p/n}-1\right)\right| \le \frac{2p}{n}\,e^{2p/n}. \tag{24}$$

Using Lemma 2, (6), (7), (11) and (19)–(24), we obtain

$$\begin{aligned} w_q(x)\left|A_n\left((t-x)e^{pt};x\right)\right| &= \frac{xe^{-qx}}{1+\sinh nx}\left|e^{p/n}\left(\cosh\left(e^{p/n}nx\right)-\sinh\left(e^{p/n}\,nx\right)\right)\right. \\ &\quad \left. + \left(e^{p/n}-1\right)\sinh\left(e^{p/n}nx\right)-1\right| \\ &\le \frac{xe^{-qx}}{1+\sinh nx}\left(e^{p/n}+\frac{p}{n}\,e^{p/n}\sinh\left(e^{p/n}\,nx\right)+1\right) \\ &\le 2(e^p+1)\,\frac{x}{e^{nx}}+2pe^p\,\frac{x}{n}\,e^{-(q-p_n)x} \le M_{p,q}\,\frac{x+1}{n}\,, \end{aligned}$$

$$\begin{aligned} w_q(x)A_n\left((t-x)^2e^{pt};x\right) &\le \frac{x^2e^{-qx}}{1+\sinh nx}\left(2e^{p/n}+1+\left(e^{p/n}nx-1\right)\sinh\left(e^{p/n}nx\right)\right)\\ &\qquad + 2e^{p/n}\,\frac{x}{n}\,e^{-(q-p_n)x}\\ &\le 4(2e^p+1)\,\frac{1}{n^2}+\frac{p}{n}\,e^{p/n}\,x^2\,e^{-(q-p_n)x}\\ &\le 4(2e^{2p}+1)\,\frac{x+1}{n}+\frac{pe^p}{n}\,\frac{2}{\left(q-p_{n_0}\right)^2}\le M_{p,q}\,\frac{x+1}{n}, \end{aligned}$$

for all $x \ge 0$ and $n \ge n_0$. Hence the proof of (14)–(17) is completed.

Similarly, using (14), (16) and (17), we derive (18) from (8).

Lemma 5. *If $f \in C_p$ with some $p > 0$ and if q, n_0 satisfy the assumptions of Lemma 4, then*

$$\|A_n(f;\cdot)\|_{C_q} \le 2\|f\|_{C_p}, \tag{25}$$

$$\|B_n(f;\cdot)\|_{C_q} \le 2(p+1)e^{2p}\|f\|_{C_p}, \tag{26}$$

for all $n \ge n_0$.

Proof. From (1)–(4) follows

$$\|A_n(f;\cdot)\|_{C_q} \le \|f\|_{C_p}\,\left\|A_n(e^{pt};\cdot)\right\|_{C_q},$$

$$\|B_n(f;\cdot)\|_{C_q} \le \|f\|_{C_p}\,\left\|B_n(e^{pt};\cdot)\right\|_{C_q},$$

for $n \in \mathbf{N}$, which by (14) and (15) imply the desired inequalities (25) and (26).

3. APPROXIMATION THEOREMS

In this part we shall give two theorems on the degree of approximation of functions belonging to the space C_p by the operators A_n and B_n in the norm of C_q, $q > p$. Since the proofs of these theorems for the operators B_n are similar to the proofs for A_n, we shall prove our results only for the operators A_n.

Theorem 1. *Suppose that $g \in C_p^1 := \left\{f \in C_p : f' \in C_p\right\}$ with some $p > 0$, $q > p$ and n_0 is a fixed natural number satisfying the condition* (13)*. Then there exists a positive constant $M_{p,q}$ depending only on p, q such that*

$$w_q(x)\left|A_n(g;x)-g(x)\right| \le M_{p,q}\,\|g'\|_{C_p}\left(\frac{x+1}{n}\right)^{1/2}, \tag{27}$$

$$w_q(x)\left|B_n(g;x)-g(x)\right| \le M_{p,q}\,\|g'\|_{C_p}\left(\frac{x+1}{n}\right)^{1/2},$$

for all $x \geq 0$ *and* $n \geq n_0$.

Proof. Let $x \geq 0$ be a fixed point. If $g \in C_p^1$, then

$$g(t) - g(x) = \int_x^t g'(u)\,\mathrm{d}u, \quad t \geq 0,$$

and by (3) and (6) for every $n \in \mathbf{N}$ we have

$$A_n\big(g(t);x\big) - g(x) = A_n\bigg(\int_x^t g'(u)\,\mathrm{d}u;\, x\bigg).$$

Since

$$\bigg|\int_x^t g'(u)\,\mathrm{d}u\bigg| \leq \|g'\|_{C_p}\bigg|\int_x^t \frac{1}{w_q(u)}\,\mathrm{d}u\bigg| \leq \|g'\|_{C_q}\,(e^{pt}+e^{px})|t-x|,$$

we get

$$\begin{aligned} w_q(x)\big|A_n(g(t);x) - g(x)\big| &\leq w_q(x)A_n\bigg(\bigg|\int_x^t g'(u)\,\mathrm{d}u\bigg|;x\bigg) \\ &\leq \|g'\|_{C_p}\, w_q(x)\,\Big(A_n\big(|t-x|e^{pt};x\big) + e^{px}\,A_n\big(|t-x|;x\big)\Big). \end{aligned}$$

Using the HÖLDER inequality and (6), (9), (14) and (17), we get

$$A_n\big(|t-x|;x\big) \leq 2\,\Big(A_n\big((t-x)^2;x\big)\Big)^{1/2}\Big(A_n(1;x)\Big)^{1/2} \leq 8\left(\frac{x+1}{n}\right)^{1/2},$$

$$\begin{aligned} w_q(x)A_n\big(|t-x|e^{pt};x\big) \;&\leq\; 2w_q(x)\Big(A_n\big((t-x)^2e^{pt};x\big)\Big)^{1/2}\Big(A_n\big(e^{pt};x\big)\Big)^{1/2} \\ &\leq\; M_{p,q}\left(\frac{x+1}{n}\right)^{1/2}, \end{aligned}$$

for all $n \geq n_0$. Summing up, we obtain the desired inequality (27).

Theorem 2. *Suppose that* $f \in C_p$, *with some* $p > 0$, *and the numbers* q *and* n_0 *satisfy the assumptions of Theorem 1. Then there exists a positive constant* $M_{p,q}$ *depending only on* p *and* q *such that for all* $x \geq 0$ *and* $n \geq n_0$ *hold the following inequalities*

$$w_q(x)\big|A_n(f;x) - f(x)\big| \leq M_{p,q}\,\omega\left(f, C_p; \left(\frac{x+1}{n}\right)^{1/2}\right), \tag{28}$$

$$w_q(x)\big|B_n(f;x) - f(x)\big| \leq M_{p,q}\,\omega\left(f, C_p; \left(\frac{x+1}{n}\right)^{1/2}\right).$$

Proof. Let f_h be the STEKLOV mean of $f \in C_p$, i.e.

$$f_h(x) = \frac{1}{h} \int_0^h f(x+u)\,\mathrm{d}t \qquad x \geq 0,\ h > 0.$$

For $x \geq 0$ and $h > 0$ we have

$$f_h(x) - f(x) = \frac{1}{h} \int_0^h \big(f(x+u) - f(x)\big)\,\mathrm{d}t, \quad f_h'(x) = \frac{1}{h}\big(f(x+h) - f(x)\big),$$

which imply $f_h \in C_p^1$ and by (2)

$$\|f_h - f\|_{C_p} \leq \omega(f, C_p; h), \tag{29}$$

$$\|f_h'\|_{C_p} \leq h^{-1}\omega(f, C_p; h). \tag{30}$$

It is obvious that for every $x \geq 0$, $n \in \mathbf{N}$, $h > 0$ and $q > p$ holds

$$\begin{aligned} w_q(x)\big|A_n(f;x) - f(x)\big| \ &\leq \ w_q(x)\Big(\big|A_n(f - f_n; x)\big| \\ &\quad + \ \big|A_n(f_h;x) - f_h(x)\big| + \big|f_h(x) - f(x)\big|\Big). \end{aligned}$$

Using Lemma 5 and (29), we get

$$w_q(x)\big|A_n(f - f_h; x)\big| \leq 2\|f - f_h\|_{C_p} \leq 2\omega(f, C_p; h)$$

for $x \geq 0$, $h > 0$ and $n \geq n_0$. By Theorem 1 and (30) we have

$$\begin{aligned} w_q(x)\,\big|A_n(f_h;x) - f_h(x)\big| &\leq M_{p,q}\,\|f_h'\|_{C_p}\left(\frac{x+1}{n}\right)^{1/2} \\ &\leq M_{p,q}\,h^{-1}\omega(f, C_q; h)\left(\frac{x+1}{n}\right)^{1/2}, \quad \text{for } \ x \geq 0,\ n \geq n_0 \ \text{ and } \ h > 0. \end{aligned}$$

Combinig these, we obtain

$$w_q(x)\,\big|A_n(f;x) - f(x)\big| \leq \omega(f, C_p; h)\left(3 + M_{p,q}h^{-1}\left(\frac{x+1}{n}\right)^{1/2}\right)$$

for every $x \geq 0$, $n \geq n_0$ and $h > 0$. Setting $h = \left(\dfrac{x+1}{n}\right)^{1/2}$, for every fixed $x \geq 0$ and $n \geq n_0$, we obtain (28).

From Theorem 2 we can derive the following two corollaries.

Corollary 1. *If* $f \in C_p$ *with some* $p > 0$, *then*

$$\lim_{n\to+\infty} A_n(f;x) = f(x), \qquad \lim_{n\to+\infty} B_n(f;x) = f(x),$$

for all $x \geq 0$. *Moreover, the convergence holds uniformly on every interval* $[0, a]$, $a > 0$.

Corollary 2. *Let* $f \in \mathrm{Lip}\,(C_p, \alpha)$ *with some* $p > 0$ *and* $0 < \alpha \leq 1$ *and let* $q > p$. *Then there exists a positive constant* $M_{p,q}$ *depending only on* p, q *such that*

$$w_q(x)\left|A_n(f;x) - f(x)\right| \leq M_{p,q}\left(\frac{x+1}{n}\right)^{\alpha/2},$$

$$w_q(x)\left|B_n(f;x) - f(x)\right| \leq M_{p,q}\left(\frac{x+1}{n}\right)^{\alpha/2},$$

for all $x \geq 0$ and $n > p(\ln(q/p))^{-1}$.

REFERENCES

1. M. Becker, D. Kucharski, R. J. Nessel: *Global approximation theorems for Szasz–Mirakjan operators in exponential weight spaces.* Linear Spaces and Approximation (Proc. Conf. Oberwolfach, 1977), Birkhäuser Verlag, Basel.
2. A. F. Timan: *Theory of Approximation of Functions of a Real Variable.* New York 1963.

Institute of Mathematics,
Technical University of Poznań,
Piotrowo 3A,
60–965 Poznań, Poland

(Received June 28, 1995)