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SOME GENERALISATIONS OF MEASURABLE FUNCTIONS

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We define the notion of pseudosubmeasure as a generalisation of the submeasure notion [2], and we study some properties of the topological ring of sets defined by that. Using families of pseudosubmeasures and the associated topological rings, the pseudosubmeasurable function concept is then defined. Finally, the convergence in measure, almost everywhere convergence and almost uniform convergence are generalized to the sequences of functions with values in pseudometric space.

1. INTRODUCTION

This Note conserves the terminology and notation from [**2**].

Let $\mathcal{S}$ be a ring (or algebra) of subsets of a fixed set S. A mapping $\eta:\ \mathcal{S}\to\mathbf{R}_+$ is said to be a submeasure if:

(S_1) $\quad \eta(\emptyset)=0,$

(S_2) $\quad E\subset F\ \Rightarrow\ \eta(E)\le\eta(F):\quad E,F\in\mathcal{S},$

(S_3) $\quad \eta(E\cup F)\le\eta(E)+\eta(F);\quad E,F\in\mathcal{S}.$

In the sequel we will generalize this notion.

Let D be an ordered set with the smallest element d_0, in which it was defined a mapping: $(d_1,d_2)\to d_1+d_2$ with the properties:

(P_1) $\quad d_0+d=d+d_0=d;\quad \forall\ d\in D,$

(P_2) $\quad d_1+d_2=d_2+d_1;\quad \forall\ d_1,d_2\in D,$

(P_3) $\quad d_1\le d_2\ \Rightarrow\ d+d_1\le d+d_2;\quad \forall\ d\in D.$

There exists a subset $D_1\subseteq D$, left directed so that:

(P_4) $\quad \forall\ d\in D_1,\ \ \exists\ d_1\in D_1\ $ so that $\ d_1+d_1\le d.$

Definition 1.1. *A pseudometric on a set* X *is a* D*-valued function* $\rho:\ X\times X\to D$ *so that*:

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(i) $\rho(x,y) = d_0 \Leftrightarrow x = y$,

(ii) $\rho(x,y) = \rho(y,x); \quad x,y \in X$,

(iii) $\rho(x,y) \le \rho(x,z) + \rho(z,y); \quad x,y,z \in X$.

A set X together with a pseudometric ρ is called a pseudometric space and is denoted by (X,ρ,D).

REMARK 1.2. The family $\{B(x_0,d)\}_{d\in D_1}$ where $B(x_0,d) = \{x \in X;\ \rho(x_0,x) \le d\}$ constitutes a base of neighbourhoods of $x_0 \in X$.

Theorem 1.3. *Every uniform space $(X,\mathcal{U})$ is pseudometrizable.*

Proof. Let D be the family of subsets of cartesian square $X \times X$ which contains the diagonal Δ_X. The order relation is inclusion $\subseteq$, and the smallest element is Δ_X. If $V_1, V_2 \in D$, we define $+ : D \times D \to D$ using $V_1 + V_2 = (V_1 \circ V_2) \cup (V_2 \circ V_1)$.

If $\Delta_X \subset V_1$, $\Delta_X \subset V_2$, it results $\Delta_X \subset V_1 \circ V_2$. From $\Delta_X \circ V = V$ it results that $\Delta_X + V = V + \Delta_X = V$. Also:

$$V_1 + V_2 = (V_1 \circ V_2) \cup (V_2 \circ V_1) = (V_2 \circ V_1) \cup (V_1 \circ V_2) = V_2 + V_1.$$

Let $V_1 \subseteq V_2$ and V arbitrary. Let $(x,y) \in V_1$ and $(y,z) \in V$; therefore $(x,z) \in V \circ V_1$. Since $V_1 \subseteq V_2$, $(x,y) \in V_2$, it follows that $(x,z) \in V_1 \circ V$. Let $(x,y) \in V$ and $(y,z) \in V_1$; it result that $(x,z) \in V_1 \circ V$; but $(y,z) \in V_2$, hence $(x,z) \in (V_2 \circ V_1)$ and $V_1 + V \subset V_2 + V$.

We consider $D_1 \subseteq D$, where D_1 is the set of symmetric entourages.

The map $\rho : X \times X \to D$; $\rho(x,y) = \{(x,y),(y,x)\} \cup \Delta_X$ is a pseudometric. Indeed, $\rho(x,y) = \Delta_X$ iff $x = y$, and $\rho(x,y) = \rho(y,x)$.

Moreover:

$$\begin{aligned}
\rho(x,z) + \rho(z,y) &= \{(x,z),(z,x)\} \cup \Delta_X + \{(z,y),(y,z)\} \cup \Delta_X \\
&= \{(x,z),(z,x),(x,y),(z,y),(y,z),(y,x)\} \cup \Delta_X \\
&\supseteq \{(x,y),(y,x)\} \cup \Delta_X = \rho(x,y).
\end{aligned}$$

Since D_1 is the set of symmetric entourages, it results that $x \in B(x_0,V)$ iff $(x,x_0) \in V_0$. Indeed, $x \in B(x_0,V)$ imply $\rho(x_0,x) = \{(x_0,x),(x,x_0)\} \cup \Delta_X \subseteq V$ hence $(x,x_0) \in V$. Conversely, $(x_0,x) \in V$ imply $(x,x_0) \in V$ hence $\rho(x,x_0) \subseteq V$.

Thus the uniform space $(X,\mathcal{U})$ and the pseudometric space (X,ρ,D) are topologically equivalent.

Definition 1.4. *A pseudosubmeasure on a ring $\mathcal{S} \subset \mathcal{P}(S)$ is a mapping $\gamma : \mathcal{S} \to D$ so that*:

(S_1) $\gamma(\emptyset) = d_0$,

(S_2) $E \subseteq F \Rightarrow \gamma(E) \le \gamma(F); \quad E,F \in \mathcal{S}$,

(S_3) $\gamma(E \cup F) \le \gamma(E) + \gamma(F); \quad E,F \in \mathcal{S}$.

If γ has the property that $\gamma(A) = d_0 \Rightarrow A = \emptyset$, then the mapping

$$\rho : \mathcal{S} \times \mathcal{S} \to D;\ \rho(A,B) = \gamma(A \,\Delta\, B)$$

is a pseudometric on $\mathcal{S}$ invariant to translation Δ (symmetric difference).

2. THE TOPOLOGICAL RING DEFINED BY A FAMILY OF PSEUDOSUBMEASURES

Let γ be a pseudosubmeasure on $\mathcal{S}$.

Theorem 2.1. *The family $\Omega = \{\mathcal{U}_d\}_{d\in D_1}$, where $\mathcal{U}_d = \{A \in \mathcal{S};\ \gamma(A) \le d\}$ constitutes a base of neighbourhoods of $\emptyset$ for a Fréchet–Nikodym topology $\tau(\gamma)$ so that $\mathcal{S}(\gamma) = (\mathcal{S}, \Delta, \cup, \tau(\gamma))$ is a uniformizable topological ring.*

Proof. For every $d \in D_1$, there exists $d_1 \in D_1$ so that $d_1 + d_1 \le d$.

Let $E, F \in \mathcal{U}_{d_1}$. We have:

$$\gamma(E\Delta F) \le \gamma(E \cup F) \le \gamma(E) + \gamma(F) \le d_1 + d_1 \le d.$$

It results $\mathcal{U}_{d_1} \circ \mathcal{U}_{d_1} \subseteq \mathcal{U}_d$. Also $\mathcal{U}_d \overset{\circ}{\cap} \mathcal{U}_d \subseteq \mathcal{U}_d$ and for $E \in \mathcal{S}$, $E \overset{\circ}{\cap} \mathcal{U}_d \subseteq \mathcal{U}_d$. The set $\{\mathcal{W}_d\}_{d\in D_1}$, where $\mathcal{W}_d = \{(A, B) \in \mathcal{S} \times \mathcal{S};\ \gamma(A\Delta B) \le d\}$ is the base of an entourage filter of the ring $\mathcal{S}$.

Corollary 2.2. *Let $\Gamma = \{\gamma_i\}_{i\in I}$ be a family of pseudosubmeasures on $\mathcal{S}$ and let the family $\Omega_\Gamma = \{\mathcal{V}_{K,d};\ K = \text{finite} \subseteq I,\ d \in D_1\}$ where $\mathcal{V}_{K,d} = \{A \in \mathcal{S};\ \gamma_i(A) \le d;\ i \in K\}$. Then there exists a FN–topology $\tau(\Gamma)$ on $\mathcal{S}$ so that $\mathcal{S}(\Gamma) = (\mathcal{S}, \Delta, \cap, \tau(\Gamma))$ is a topological ring.*

Theorem 2.3. *The topological ring $\mathcal{S}(\gamma) = (\mathcal{S}, \Delta, \cap, \tau(\gamma))$ iz separated iff: $\forall d_1 \in D$, $\forall d \in \gamma(\mathcal{S})$, $d \le d_1$ implies $d = d_0$.*

(We denote $\gamma(\mathcal{S}) = \{d \in D,\ \exists A \in \mathcal{S};\ \gamma(A) = d\}$.)

Proof. Suppose that $\mathcal{S}$ with the topology $\tau(\gamma)$ is separated and that there exists $d \in \gamma(\mathcal{S})$ so that $d \le d_1$, $\forall d_1 \in D_1$. Then there is $E \in \mathcal{S}$ with $\gamma(E) = \gamma(E\Delta\emptyset) = d \le d_1$ and E belongs to any neighbourhood of $\emptyset$. Because the space is separated, it results that $E = \emptyset$, i.e. $d = d_0$.

Conversely, suppose that conditions of theorem hold and $\tau(\gamma)$ is not separated, hence there exists $A \ne \emptyset$ so that $\forall d \in D_1$ with $d_1 + d_1 \le d$. We have $\mathcal{A} \cap \mathcal{B} = \emptyset$, where $\mathcal{A} = \{E \in \mathcal{S};\ \gamma(E) \le d_1\}$, $\mathcal{B} = \{E \in \mathcal{S};\ \gamma(E\Delta A) \le d_1\}$. Then there is $C \in \mathcal{A} \cap \mathcal{B}$ so that

$$\begin{aligned} d &= \gamma(E) = \gamma\big((E\Delta C)\,\Delta\,(C\Delta\emptyset)\big) \\ &\le \gamma\big((E\Delta C) \cup (C\Delta\emptyset)\big) \le \gamma(E\Delta C) + \gamma(C) \le d_1 + d_1 \le d, \end{aligned}$$

that contradicts the hypothesis.

3. ABSOLUTE CONTINUITY

Suppose that the ordered groupoid $(D, +)$ has the properties P_1, P_2, P_3, $P_4{}'$ and P_5, where

($P_4{}'$) For any $d \in D$ and for any $n \in \mathbf{N}$, there exists $c \in D$ so that $c_1 + c_2 + \cdots + c_n = d$, where $c_i = c$, $i = \overline{1, n}$. The grupoid $(D, +)$ with this property is called complet.

(P_5) For any $M \subset D$ there exists $\inf f(M)$.

Theorem 3.1. *The couple $(D, +)$ is a topological complete groupoid.*

Proof. The element $c \in D$ from ($P_4{}'$) is denoted $c = (1/n)d$. For every $r \in Q,\ r \geq 0$ and every $d \in D,\ rd \in D$. For $a \in D$ set $L_a = \{ra;\ r \in Q,\ r \geq 0\}$ is also a complete groupoid with unit d_0 and the element a is called the generator of the groupoid. The mapping $\varphi : L_a \to Q,\ \varphi(ra) = r$ is a monomorphism for the groupoid L_a to the groupoid $(Q, +)$. The natural topology of Q induces on $\varphi(L_a)$ a topology which defines a topology on L_a by φ^{-1}. The subset $V_{d_0}(\varepsilon, L_a) = \{x \in L_a;\ x = ra;\ 0 < r < \varepsilon\}$ is called spherical neighbourhood of radius in L_a for the unit element $d_0 \in D$. If $b \in L_a$, then $L_a = L_b$. For $b \in L_a,\ V_{d_0}(\varepsilon, L_b) = V_{d_0}(\varepsilon, L_a)$. Then it results that the spherical neighbourhood set in L_a is not depending on the choice of the generator from L_a.

In the sequel we define a base of neighbourhoods of $d_0 \in D$ so that: $V \subset D$ is called neighbourhood for $d_0 \in D$ if $\forall a \in D,\ V \cap L_a$ is a spherical neighbourhood in L_a for $d_0 \in D$. So we obtain a topology $\Im_c$ on the complete groupoid with unit $(D, +)$.

Let $\gamma : \mathcal{S} \to D$ be a pseudosubmeasure. The mapping

$$\gamma^* : \mathcal{P}(S) \to D,\ \gamma^*(E) = \inf\{\gamma(A);\ E \subseteq A \in \mathcal{S}\},$$

$E \subseteq S$ is called the Jordan extension of γ. It is verified that $\gamma*$ is a pseudosubmeasure on the algebra $\mathcal{P}(S)$.

Let $a \in D$ fixed, $\Gamma = \{\gamma_i\}_{i \in I}$ a family of pseudosubmeasure on $\mathcal{S}$ and let the family:

$$\Omega^*_\Gamma = \left\{\mathcal{V}^*_{K,d};\ K = \text{finite} \subseteq I,\ d = ra,\ 0\langle r,\ r \in Q\right\},$$

where

$$\mathcal{V}^*_{K,d} = \left\{E;\ E \subseteq S;\ \gamma^*_i(E) \leq d,\ i \in K\right\}.$$

Then there exists a unique topology $\tau(\Gamma)$ on $\mathcal{P}(S)$ so that

$$\mathcal{P}(S)(\Gamma) = (\mathcal{P}(S), \Delta, \cap, \tau(\Gamma))$$

is a topological ring and Ω^*_Γ is a base of neighbourhood of $\emptyset$ for this topology.

A set $N \subseteq S$ is Γ-negligible if for every $i \in I,\ \gamma^*_i(N) = d_0$. We denote $\mathcal{N}_\Gamma = \{N;\ N \subseteq S :\ N \text{ is } \Gamma\text{-negligible}\}$.

Definition 3.2. *Let $\Gamma = \{\gamma_i\}_{i \in I}$ and $\Gamma' = \{\gamma'_j\}_{j \in I}$ be two families of pseudosubmeasures on $\mathcal{S}$. Γ' is absolutely continuous with respect to Γ, and denote $\Gamma' \ll \Gamma$ if for every $j \in J$ we have:*

$$\lim_{E \to \emptyset \text{ in } \mathcal{P}(S)(\Gamma)} {\gamma'_j}^*(E) = d_0$$

(the limit is taken on the $\Im_c$*-topology from* $(D, +)$ *).*

4. PSEUDOSUBMEASURABLE FUNCTIONS. TYPES OF CONVERGENCE

Let $\Gamma = \{\gamma_i : \mathcal{S} \to D\}_{i \in I}$ be a family of pseudosubmeasures on $\mathcal{S} \subset \mathcal{P}(S)$ and let (X, ρ, D) a pseudometric space.

By generalizing the model established in **[3]** we will introduce an uniform structure on X^S in the following way:

To every $K =$ finite $\subset I$, $d \in D$, we associate the set:

$$\mathcal{W}_K(d) = \big\{(f, g) \in X^S \times X^S;\ \gamma_i^*\{s \in S;\ \rho(f(s), g(s))\rangle d\} \le d, i \in K\big\}.$$

Theorem 4.1. *The family* $\{\mathcal{W}_K(d);\ d = D_1,\ K = \text{finite} \subseteq I\}$ *forms a base for uniform structure* U_Γ *on* X^S.

Proof. Let $d \in D_1$ and $K = \text{finite} \subset I$. It is clear that $(f, f) \in \mathcal{W}_K(d)$ for any $f \in X^S$ and that $\mathcal{W}_K(d)$ is symmetrical. Let $d_1 \in D$ so that $d_1 + d_1 \le d$ and let $(f, g) \in \mathcal{W}_K(d_1) \circ \mathcal{W}_K(d_1)$. Then we have $h \in X^S$ so that $(f, h), (h, g) \in \mathcal{W}_K(d_1)$ and:

$$\begin{aligned}&\{s \in S; \rho\big(f(s), g(s)\big) > d\}\\ &\quad \subseteq \{s \in S; \rho\big(f(s), h(s)\big) > d_1\} \cup \{s \in S;\ \rho\big(h(s), g(s)\big) > d_1\}.\end{aligned}$$

Thus: $\gamma_i^*\{s \in S;\ \rho\big(f(s), g(s)\big)\rangle d\} \le d_1 + d_2 \le d,\ i \in K$, which shows that $(f, g) \in \mathcal{W}_K(d)$. So $\mathcal{W}_K(d_1) \circ \mathcal{W}_K(d_1) \subseteq \mathcal{W}_K(d)$.

Finally, let $d_1, d_2 \in D_1$ and K_1, K_2 finite $\subset I$. If $d' \in D_1$, $d' \le d_1$ and $d' \le d_2$, we have:

$$\mathcal{W}_{K_1 \cup K_2}(d') \subseteq \mathcal{W}_{K_1}(d_1) \cap \mathcal{W}_{K_2}(d_2).$$

We denote: $X^S(\Gamma) = (X^S, U_\Gamma)$.

Remark 4.2. If $\Gamma' \langle\langle \Gamma$, then $U_{\Gamma'} \subseteq U_\Gamma$.

The map $f \in X^S$ is a $\mathcal{S}$-step function if there exists $x_i \in X,\ E_i \in \mathcal{S}$, $i = 1, 2 \ldots n,\ x_i \ne x_j,\ E_i \cap E_j = \emptyset,\ i \ne j,\ S = \cup_{i=1}^n E_i$ so that $\forall s \in E_i$ imply $f(s) = x_i,\ i = 1, 2, \ldots, n$.

The space of $\mathcal{S}$-step functions will be denoted by $\mathcal{E}(\mathcal{S}, (X, \rho)) \subset X^S$.

Definition 4.3. *The function* $f \in X^S$ *is* Γ*-pseudosubmeasurable if* f *belongs to the adherence of* $\mathcal{E}(\mathcal{S}, (X, \rho))$ *in* $X^S(\Gamma)$.

We denote by $\mathcal{M}(\mathcal{S}, \Gamma, (X, \rho))$ *the set of all these functions.*

Remark 4.4. If $\Gamma' \langle\langle \Gamma$, then $\mathcal{M}(\mathcal{S}, \Gamma, (X, \rho)) \subseteq \mathcal{M}(\mathcal{S}, \Gamma', (X, \rho))$.

Definition 4.5. *Let* $\{f_\alpha\}$ *be a generalized sequence in* X^S *and* $f \in X^S$.

a) *If* $f_\alpha \to f$ *in* $X^S(\Gamma)$, *then* $\{f_\alpha\}$ *converges to* f *in* Γ*-pseudosubmeasures and we denote* $f_\alpha \xrightarrow{\Gamma} f$.

b) *If there exists* $A \in \mathcal{N}_\Gamma$ *such that* $f_\alpha \to f(s)$ *for every* $s \in S - A$ *in* (X, ρ, D), *then* $\{f_\alpha\}$ *converges* Γ*-almost everywhere to* f *and we denote* $f_\alpha \to f$ *a.e.* (Γ).

c) *If there exists a generalized sequence* $\{A_\beta\}$ *in* $\mathcal{P}(S)$ *such that* $A_\beta \setminus \emptyset$ *in* $\mathcal{P}(S)(\Gamma)$ *and if for any* β, $f_\alpha(s) \to f(s)$ *uniformly on* $S - A_\beta$, *then* $\{f_\alpha\}$ *converges to* $f\Gamma$*-almost uniformly and we denote* $f_\alpha \to f$ *a.u.*

Theorem 4.6.

i) $$f_\alpha \to f \ a.u.(\Gamma) \Rightarrow f_\alpha \xrightarrow{\Gamma} f.$$

ii) $$f_\alpha \to f \ a.u.(\Gamma) \Rightarrow f_\alpha \to f \ a.e.(\Gamma)$$

If $\Gamma' \langle\langle \Gamma$ *then*:

iii) $$f_\alpha \to f \ a.e.(\Gamma) \Rightarrow f_\alpha \to f \ a.e.(\Gamma')$$

iv) $$f_\alpha \to f \ a.u.(\Gamma) \Rightarrow f_\alpha \to f \ a.u.(\Gamma')$$

Proof. The implications result from Definition 4.5.

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