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HALPERN–LÄUCHLY THEOREM ON THE PRODUCT OF INFINITELY MANY TREES IN THE VERSION OF STRONGLY EMBEDDED TREES

Goran Ramović

Laver in [1] formulated a special case of Halpern–Läuchly theorem in the version of strongly embedded trees on the product of countable many trees. In this article a general version of this theorem is proved.

Laver ([**1**] the corollary of Theorem 5) formulated a special case of Halpern–Läuchly theorem in the version of strongly embedded trees on the product of countable many trees. He concentrated on the case where the countable sequence of $\langle \omega, < \omega \rangle$-trees belongs to class K_ω. In the proof of Theorem 5 an error appeared. Namely, Lema $4'$ was formulated for the countable sequence of $\langle \omega, < \omega \rangle$-trees of class K_ω and its proof was basically referred to the properties of the sequence of trees of that class. The assertion of Lemma $4'$ was then, in the proof of Theorem 5, applied to the countable sequence of $\langle \omega, < \omega \rangle$-trees which does not belong to class K_ω.

In this article, we formulate and prove the general Halpern–Läuchli theorem in the version of strongly embedded trees on the product of countably many trees. We suppose that the reader is familiar with Laver's [**1**] and Milliken's [**2**] articles where the same terminology and denotation are used.

The idea of proof of the following theorem can be explained as follows. Let f be the partitioning of $\underset{i \in \omega}{\otimes} T_i$ into $r \in \omega \setminus 1$ partitions. Surjection F from $\underset{i \in \omega}{\otimes} T'_i$ onto $\underset{i \in \omega}{\otimes} T_i$ preserves the order and the levels, shifting to more regular trees. In addition, surjection G from $\underset{i \in \omega}{\otimes} T_i \underset{j \in \omega}{\otimes} T''_{ij}$ onto $\underset{i \in \omega}{\otimes} T'_i$ preserves the order and the levels, and the sequence of ω-trees with finitely many branches per tree, restricted to the levels of an infinite subset of ω, is mapped into the sequence of strongly embedded trees. With the help of surjections F and G, partitioning $h = f \circ F \circ G$ of set $\underset{i \in \omega}{\otimes} T_i \underset{i \in \omega}{\otimes} T''_{ij}$ into r parts is defined. Afterwards, on the basis of the corollary which concludes the series of lemmas, as well as on the properties of surjections F and G, the proof of the theorem follows.

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Theorem. *Let suppose that*:

$\vec{T} = \langle T_i : i \in \omega \rangle$ *is a sequence of* $\langle \omega, < \omega \rangle$*-trees*;

$f : \bigotimes_{i \in \omega} T_i \to r$, *where* $r \in \omega \setminus 1$. *Then* $\vec{t} = \langle t_i : i \in \omega \rangle$, $\vec{S} = \langle S_i : i \in \omega \rangle$ *and* $k \in r$ *exist for which the following hold*:

(1) $\vec{t} \in \prod_{i \in \omega} T_i$;

(2) $\vec{S} = \mathrm{Str}^{\omega}(\langle T_i[t_i] : i \in \omega \rangle)$;

$f'' \bigotimes_{i \in \omega} S_i = \{k\}$.

Proof. Let us assume that the theorem has been proved for $r \leq 2$. Let $r > 2$. Function g will be defined from $\bigotimes_{i \in \omega} T_i$ into 2 in the following way. For an arbitrary $\vec{x} \in \bigotimes_{i \in \omega} T_i$ the following shall be true:

$$g(\vec{x}) = \begin{cases} 0 & \text{if } f(\vec{x}) \in r - 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\vec{t} = \langle t_i : i \in \omega \rangle$, $\vec{S} = \langle S_i : i \in \omega \rangle$ and $k \in 2$ exists to which (1), (2) and

$$g'' \bigotimes_{i \in \omega} S_i = \{k\}$$

are applicable. If $k = 0$, we shall apply an inductive assumption, but in case $k = 1$ the theorem is proved immediately.

Let us proceed to proving the theorem for $r = 2$. We can obviously still retain generality and suppose that the following applies to each $i \in \omega$:

(3) $T_i \subseteq^{<\omega} \omega$;

(4) $\forall x \in T_i \; \forall y \in^{<\omega} \omega \quad (y \subseteq x \to y \in T_i)$;

(5) $\forall x \in T_i \; \forall y \in^{<\omega} \omega \quad \big(|x| = |y| \,\&\, \forall n \in \mathrm{dom}(x)(y(n) \leq x(n)) \to y \in T_i\big)$;

(6) $\forall x \in T_i \; \forall y \in T_i \quad (x \leq_i y \leftrightarrow x \subseteq y)$.

For the reason mentioned in (6), the relation $\leq_i$ will hereinafter be denoted $\leq$. It can easily be seen that a function g_i from ω into $\omega \setminus 1$ exists for every $i \in \omega$, such that

$$\forall n \in \omega \; \forall x \in T_i(n) \big(|\mathrm{IS}(x, T_i)| \leq g_i(n)\big).$$

We call sequence $\vec{T} = \langle T_i : i \in \omega \rangle$ an $\langle \omega, \leq \vec{g} \rangle$-sequence of trees, where $\vec{g} = \langle g_i : i \in \omega \rangle$. The sequence of $\langle \omega, < \omega \rangle$-trees wil be denoted $\vec{T}' = \langle T'_i : i \in \omega \rangle$ to which (3)–(6), as well as for every $i \in \omega$, the following applies:

$$\forall n \in \omega \; \forall x \in T'_i(n) \big(|\mathrm{IS}(x, T'_i)| = g_i(n)\big).$$

We call sequence $\vec{T}' = \langle T'_i : i \in \omega \rangle$ an $\langle \omega, \vec{g} \rangle$-sequence of trees.

Lemma 1. *There is a surjection* F *from* $\bigotimes_{i \in \omega} T'_i$ *onto* $\bigotimes_{i \in \omega} T_i$, *with the following properties*:

(7) $\forall \vec{x} \in \bigotimes_{i \in \omega} T'_i \; \forall \vec{y} \in \bigotimes_{i \in \omega} T'_i \; \big(\vec{x} \leq \vec{y} \to F(\vec{x}) \leq F(\vec{y})\big)$;

(8) $$\forall n \in \omega\, \forall \vec{x} \in \underset{i \in \omega}{\otimes} T_i' \left(\vec{x} \in \prod_{i\in\omega} T_i'(n) \leftrightarrow F(\vec{x}) \in \prod_{i\in\omega} T_i(n)\right);$$

(9) $$\forall \vec{S'} \in \mathrm{Str}^\omega(\vec{T'}) \exists \vec{S} \in \mathrm{Str}^\omega(\vec{T}) \left(F'' \underset{i \in \omega}{\otimes} S_i' = \underset{i \in \omega}{\otimes} S_i\right).$$

Proof. For each $i \in \omega$ surjection F_i from T_i' onto T_i is defined so that:

$$\forall x \in T_i'\ \forall y \in T_i \Big(F_i(x) = y \leftrightarrow |x| = |y|\ \&$$
$$\forall\, n \in \mathrm{dom}(x)\big(y(n) = \min(x(n),\, |\mathrm{JS}(y \uparrow n, T_i)| - 1)\big)\Big).$$

It can be easily be checked that for each $i \in \omega$ the following holds:

$$\forall x \in T_i'\ \forall y \in T_i'\ \left(x \le y \to F_i(x) \le F_i(y)\right);$$
$$\forall n \in \omega\ \forall x \in T_i'\ \left(x \in T_i'(n) \leftrightarrow F_i(x) \in T_i(n)\right);$$
$$\forall S' \in \mathrm{Str}^\omega(T_i')\, \exists S \in \mathrm{Str}^\omega(T_i)\, (F_i'' S' = S).$$

We can now define surjection F from $\underset{i \in \omega}{\otimes} T_i'$ onto $\underset{i \in \omega}{\otimes} T_i$ with the help of the condition: $\forall \vec{x} \in \underset{i \in \omega}{\otimes} T_i' \,\forall \vec{y} \in \underset{i \in \omega}{\otimes} T_i \left(F(\vec{x}) = \vec{y} \leftrightarrow \forall i \in \omega \left(F_i(x_i) = y_i\right)\right)$.

It can easily be checked that F fulfils the conditions $(7) - (9)$. □

Now, for every $j \in \omega$, a unique element of set ${}^j 1$ as $0^{(j)}$ will be denoted. In addition, a T_{ij}'' will denote tree $T_i'[0^{(j)}]$ for every $i, j \in \omega$.

Lemma 2. *There is a surjection G from $\underset{i \in \omega}{\otimes} \underset{j \in \omega}{\otimes} T_{ij}''$ onto $\underset{i \in \omega}{\otimes} T_i'$ with the following properties:*

(10) $$\forall \vec{x} \in \underset{i \in \omega}{\otimes} \underset{j \in \omega}{\otimes} T_{ij}''\ \forall \vec{y} \in \underset{i \in \omega}{\otimes} \underset{j \in \omega}{\otimes} T_{ij}''\ \left(\vec{x} \le y \to G(\vec{x}) \le G(\vec{y})\right);$$

(11) $$\forall n \in \omega\, \forall \vec{x} \in \underset{i \in \omega}{\otimes} \underset{j \in \omega}{\otimes} T_{ij}'' \left(\vec{x} \in \prod_{i\in\omega}\prod_{j\in\omega} T_{ij}''(n) \leftrightarrow G(\vec{x}) \in \prod_{i\in\omega} T_i'(n)\right);$$

(12) $$\forall \vec{x}\, \forall \vec{S}\, \forall A\, \exists \vec{y}\, \exists \vec{R} \Big(\vec{x} \in \underset{i \in \omega}{\otimes} \underset{j \in \omega}{\otimes} T_{ij}''\ \&\ \forall i \in \omega\, \forall j \in \omega\, \forall k \in \omega\, \forall l \in \omega$$

$\exists m \in \omega\ (|x_{ij}| < |x_{kl}| \to |x_{im}| = |x_{kl}|)\ \&\ \forall i \in \omega\, \forall j \in \omega\ (S_{ij}$ is a downwards closed ω-subtree of the tree T_{ij}'' with $g_i(|x_{ij}|)$ branches, which includes the set $\mathrm{IS}(x_{ij}, T_{ij}''))\ \&\ \forall i \in \omega\, \forall j \in \omega\ (0^{(j)} \le x_{ij})\ \&\ A = \{|x_{ij}| : i \in \omega\ \&\ j \in \omega\} \to \vec{y} \in \prod_{i\in\omega} T_i'\ \&\ \vec{R} \in$

$$\mathrm{Str}^\omega(\langle T_i'[y_i] : i \in \omega\rangle)\ \&\ G'' \underset{i \in \omega}{\otimes}^A \underset{j \in \omega}{\otimes}^A S_{ij} = \underset{i \in \omega}{\otimes} R_i\Big).$$

Proof. For every $i \in \omega$ we define surjection G_i from $\underset{i \in \omega}{\otimes} T_{ij}''$ onto T_i' in the following way. For $n \in \omega$ and $\vec{x} \in \prod_{j\in\omega} T_{ij}''(n)$ the element $G_i(\vec{x})$ is determined by the following conditions:

$$G_i(\vec{x}) \in T_i'(n); \quad \forall m \in n \left(G_i(\vec{x})(m) = \sum_{j \le m} {}_{\mathrm{mod}\ g_i(m)} x_{ij}(m)\right).$$

It could be easily checked that for each $i \in \omega$ the following is true:

$\forall \vec{x} \in \underset{j \in \omega}{\otimes} T''_{ij} \, \forall \vec{y} \in \underset{j \in \omega}{\otimes} T''_{ij} \; (\vec{x} \leq \vec{y} \to G_i(\vec{x}) \leq G_i(\vec{y}));$

$\forall n \in \omega \, \forall \vec{x} \in \underset{i \in \omega}{\otimes} T''_{ij} \, \Big(\vec{x} \in \prod_{j \in \omega} T''_{ij}(n) \leftrightarrow G_i(\vec{x}) \in T'_i(n)\Big);$

$\forall \vec{x} \, \forall \vec{S} \, \forall A \exists R \Big(\vec{x} \in \prod_{j \in \omega} T''_{ij} \,\&\, \forall j \in \omega \;$ (S_{ij} is a downwards closed ω-subtree of the tree T''_{ij} with $g_i(|x_{ij}|)$ branches, which includes the set $\mathrm{IS}\,(x_{ij}, T''_{ij})) \,\&\, \forall j \in \omega (0^{(j)} \leq x_{ij}) \,\&\, A = \{|x_{ij} : j \in \omega\} \to y \in T'_i \,\&\, R \in \mathrm{Str}^{\omega}(T'_i) \,\&\, G''_i \underset{j \in \omega}{\otimes^A} S_{ij} = R\Big).$

Let us suppose that $\vec{x} \in \underset{i \in \omega}{\otimes} \underset{j \in \omega}{\otimes} T''_{ij}$. We shall determine $G(\vec{x}) = \langle G_i(\langle x_{ij} : j \in \omega\rangle) : i \in \omega\rangle$. Function G defined in this way, represents a surjection from $\underset{i \in \omega}{\otimes} \underset{j \in \omega}{\otimes} T''_{ij}$ onto $\underset{i \in \omega}{\otimes} T'_i$ with properties (10) – (12). $\square$

We can now define function h from $\underset{i \in \omega}{\otimes} \underset{j \in \omega}{\otimes} T''_{ij}$ into r in the following way:

$$h = f \circ F \circ G. \tag{13}$$

For each $i \in \omega$, $n \in \omega$ and $A \subseteq T_i$ we can define the set $\mathrm{proj}\,(n, A, T_i)$ as follows: $\forall B \Big(B \in \mathrm{proj}\,(n, A, T_i) \leftrightarrow B \subseteq T_i(n) \,\&\, \forall a \in A \, \exists! b \in B (a \leq b \vee b \leq a)\Big)$.

For each $d \in \omega \setminus 1$, $n \in \omega$, cofinal subset C of set $\underset{i \in d}{\otimes} T_i$ and $f : \underset{i \in d}{\otimes} T_i \to 2$, $\Phi(n, C, f)$ denotes the statement:

$\forall \vec{c} \in C \, \forall \langle B_i \in \mathrm{proj}\,(n, \mathrm{IS}\,(c_i, T_i), T_i) : i \in d\rangle \, \exists \langle b_i \in B_i : i \in d\rangle$ $(\mathrm{dom}\,(c_0) \in n \to f(\vec{b}) = 0)$.

Lemma 3. *For each $m \in \omega$, $d \in \omega \setminus 1$ and cofinal subset C of set $\underset{i \in d}{\otimes} T_i$, there exists $p = p\big(m, \underset{i \in d}{\otimes} T_i, C\big) \in \omega \setminus m$, such that:*

$\forall f : \underset{i \in d}{\otimes} T_i \to 2 \Big(\forall n \in n \in \omega \Phi(n, C, f) \to \forall n \geq p \, \exists \langle A_i \subseteq T_i(n) : i \in d\rangle$ $\big(\forall i \in d \, (A_i$ *is an m-dense set in the tree* $T_i) \,\&\, f'' \prod_{i \in d} A_i = 1\big)\Big).$

Proof. Let us suppose that the statement is not true for $m_0 \in \omega$, $d_0 \in \omega \setminus 1$ and a cofinal subset C_0 of set $\underset{i \in d_0}{\otimes} T_i$. That means that there exists a strongly increasing function g from ω into $\omega \setminus 1$ and function $f_{g(p)} : \underset{i \in d_0}{\otimes} T_i \to 2$, for each $p \geq m_0$, such that:

$\Phi\big(g(p), C_0, f_{g(p)}\big);$

$\forall \langle A_i \subseteq T_i\big(g(p)\big) : i \in d\rangle \; \Big(\forall i \in d \;$ (A_i is a m_0-dense set in the tree $T_i) \to f'' \prod_{i \in d_0} A_i \neq 1\Big).$

Let $A = (m_0 + 1) \cup \{n : \exists k \in \omega \setminus 1 \, (n = g^k(m_0))\}$. Function $f : \underset{i \in d_0}{\otimes} T_i \to 2$ may be defined as follows. For $n \in m_0 + 1$ and $\vec{t} \in \prod_{i \in d_0} T_i(n)$ we determine $f(\vec{t}) = 1$, and for $n \in A \setminus (m_0+1)$ and $\vec{t} \in \prod_{i \in d_0} T_i(n)$ we determine $f(\vec{t}) = f_n(\vec{t})$. Then there exists $\vec{t} \in \bigotimes^A_{i \in d_0} T_i$ such that:

$$\forall n \in A\, \exists m \in A \setminus n\, \exists \langle B_i \subseteq T_i[t_i]\,(m) : i \in d_0\rangle \Big(\forall i \in d_0\,(B_i \text{ is a } n\text{-dense set in the tree } T_i[t_i])\ \&\ f'' \prod_{i \in d_0} B_i = \{1\}\Big).$$

Hence, there exist $\vec{c} \in C_0$, $n \in A \setminus (\operatorname{dom}(c_0)+1)$ and $\langle B_i \in \operatorname{proj}(n, \operatorname{IS}(c_i, T_i), T_i) : i \in d_0\rangle$ such that:

$$\vec{t} \le \vec{c};$$

$$f'' \prod_{i \in d_0} B_i = \{1\}. \tag{14}$$

Condition (14) is contradictory to the definition of function f. □

Lemma 4. *Let us suppose that* $|\bigotimes_{i \in \omega} T_i| = \omega$ *and that* $f : \bigotimes_{i \in \omega} T_i \to 2$. *Then three possibilities exist:*

$$\forall m \in \omega\, \exists n \in \omega \setminus m\, \exists \langle A_i \subseteq T_i(n) : i \in \omega\rangle \Big(\forall i \le m\,(A_i \textit{ is a } m\textit{-dense set in the tree } T_i)\ \&\ \forall i > m\,(|A_i| = 1)\ \&\ f'' \prod_{i \in \omega} A_i = 1\Big); \tag{15}$$

$$\exists \vec{t} \in \bigotimes_{i \in \omega} T_i\ \forall \vec{s} \in \bigotimes_{i \in \omega} T_i\ (\vec{t} \le \vec{s} \to f(\vec{s}) = 0); \tag{16}$$

$$\exists d\, \exists \vec{t} \exists \vec{S}\, \exists A \Big(d \in \omega \setminus 1\ \&\ \vec{t} \in \prod_{i \in \omega} T_i\ \&\ \forall i \in d\,(|t_0| = |t_i|)\ \&\ \forall i \in d\,(S_i \textit{ is a downwards closed } \omega\textit{-subtree of the tree } T_i \textit{ with } g_i(|t_i|) \textit{ branches, which includes the set } \operatorname{IS}(t_i, T_i))\ \&\ \forall i \ge d\ (S_i \textit{ is a downwards closed } \omega\textit{-subtree of the tree } T_i[t_i])\ A \in [\omega]^\omega\ \&\ f'' \prod_{i \in \omega} S_i(|t_0|) = \{1\}\ \&\ \forall B\, \forall \langle s_i : i \ge d\rangle\, \forall \langle R_i : i \ge d\rangle\, \exists \langle r_i : i \ge d\rangle\ (B \in [A]^\omega\ \&\ \vec{s} \in \bigotimes^B_{i \ge d} S_i\ \&\ \forall i \ge d\,(R_i \textit{ is a downwards closed } \omega\textit{-subtree of the tree } S_i[s_i])\ \&\ \langle \bigcup_{n \in B} R_i(n) : i \ge d\rangle \in \operatorname{Str}^\omega(\langle \bigcup_{n \in A} S_i[s_i]\,(n) : i \ge d\rangle) \to \vec{r} \in \bigotimes^B_{i \ge d} R_i\ \&\ f'' \prod_{i<d} S_i(|r_d|) \times \prod_{i \ge d} \{r_i\} = \{1\})\Big). \tag{17}$$

Proof. Let us assume that neither (15) nor (16) are true. Let m be the smallest natural number for which condition (15) fails. Let us have $d = m+1$, though d can be any natural number $\ge m+1$. It can clearly be seen that $C \subseteq \bigotimes_{i \in d} T_i$ and $\langle B_i \subseteq T_i : i \ge d\rangle$ exist and they fulfil the following conditions:

C is a cofinal subset of the set $\bigotimes_{i \in d} T_i$;

$\forall i \ge d\ (B_i$ is a maximal branch in the tree $T_i)$;

$$\forall \vec{c} \in C\ \Big(f'' \prod_{i \in d} \{c_i\} \times \prod_{i \le d} B_i(|c_0|) = \{1\}\Big).$$

Let $p = p\Big(m, \bigotimes_{i \in d} T_i, C\Big)$ be a natural number whose existence has been established in Lemma 3.

Without loss of generality, we may suppose that: $\forall i \ge d\ \big(|T_i(p)| = 1\big)$.

Namely, if for any $i \geq d$ $|T_i(p)| > 1$, then we choose an arbitrary $t_i \in T_i(p)$ and change tree T_i into tree $T_i[t_i]$.

For each $\vec{c} \in C \cap \underset{i \in d}{\otimes}{}^p T_i$ and each $\vec{A} = \langle A_i \in \mathrm{proj}\,\big(p, \mathrm{IS}\,(c_i, T_i), T_i\big) : i \in d\rangle$ we may choose one $\vec{S} = \langle S_i : i \in d\rangle$ for which the following holds:

$\forall i \in d$ (S_i is a downwards closed ω-subtree of the tree T_i with $|A_i|$ branches and includes the set A_i).

Let $\langle \vec{S}_k : 1 \leq k \leq l\rangle$ be the enumeration of all the above mentioned $\vec{S}$. Let us suppose that $q \leq l$ is the bigest natural number for which sequences $\langle A_i : i \leq q\rangle$, $\langle t_i : i \leq q\rangle$ and $\langle \vec{S}_i : i \leq q\rangle$ exist with the following properties:

$$A_0 = \omega;$$
$$\vec{t}_0 = \langle \mathrm{root}\,(T_i) : i \geq d\rangle;$$
$$\vec{S}_0 = \langle T_i : i \geq d\rangle;$$
$$\forall i \leq q\ \big(A_i \in [\omega]^\omega\big);$$
$$\forall i < q\ (A_{i+1} \subseteq A_i);$$
$$\forall i \leq q\ \Big(\vec{t}_i \in \prod_{j \geq d} \bigcup_{n \in A_i} S_{ij}(n)\Big);$$
$$\forall i \leq q\ (\vec{S}_i = \langle S_{ij} : j \geq d\rangle);$$
$$\forall i < q\, \forall j \geq d\ (S_{i+1,j} \text{ is a downwards closed } \omega\text{-subtree of the tree } S_{ij});$$
$$\forall i < q\ \Big(\langle \bigcup_{n \in A_{i+1}} S_{i+1,j}(n) : j \geq d\rangle\Big) \in \mathrm{Str}^\omega\Big(\langle \bigcup_{n \in A_i} S_{ij}[t_{i+1,j}](n) : j \geq d\rangle\Big);$$
$$\forall i \leq q\, \forall n \in A_i\, \forall \vec{s}\,'' \in \prod_{j \geq d} S_{ij}(n)\, \exists \vec{s}\,' \in \prod_{j < d} S_{ij}(n)\ \big(f(\vec{s}\,'{}^\frown\vec{s}\,'') = 0\big).$$

On the basis of the previous lemma $q < l$. Now, we may choose:

$$\vec{t} = \vec{t}_{q-1};$$
$$\langle S_i : i \in d\rangle = \langle S_{qi} : i \in d\rangle;$$
$$\langle S_i : i \geq d\rangle = \langle S_{q-1,i} : i \geq d\rangle;$$
$$A = A_{q-1}.$$

$d, \vec{t}, \vec{S}$ and A chosen in this way fulfil condition (17). $\square$

Lemma 5. *Let us assume that* $\big|\underset{i \in \omega}{\otimes} T_i\big| = \omega$, $f : \underset{i \in \omega}{\otimes} T_i \to 2$, $\omega = \bigcup_{n \in \omega} E_n$ *is partitioning of set* ω *into countable many infinite partitions and* g *is strongly increasing function from* ω *into* $\omega \setminus 1$. *Then* (15) *or* (16) *hold or*

$\exists \vec{t} \in \prod_{i \in \omega} T_i\, \exists \langle S_i \subseteq T_i : i \in \omega\rangle\, \exists A \in [\omega]^\omega\ \Big(\forall i \in \omega\big(S_i$ *is a downwards closed* ω*-subtree of the tree* T_i *with* $|\mathrm{IS}\,(t_i, T_i)|$ *branches, which includes the set* $\mathrm{IS}\,(t_i, T_i)\big)\ \&\ A = \{|t_i| : i \in \omega\}\ \&\ \forall n \in \omega\, \forall j \in g(n)\, \exists i \in E_j$ ($|t_i|$ *is the n-th member of the set* A) $\&\ f''\underset{i \in \omega}{\otimes}{}^A S_i = \{1\}\Big)$.

Proof. Let us assume that neither (15) nor (16) hold for function f. We shall initiate an inductive procedure now. Let us denote function f with f_0. (17) is applicable to function f_0. We may choose $d_1 = d$ such that $\forall j \in g(0)\, \exists i \in E_j\ (i \in d_1)$. We then define function $f_1 : \bigotimes^{A_1}_{i \geq d_1} T_{1i} \to 2$, where A_1 and $\langle T_{1i} : i \geq d_1 \rangle$ are set A and sequence $\langle S_i : i \geq d \rangle$ in (17) respectively, in the following way:

$$\forall n \in A_1\, \forall \vec{s} \in \prod_{i \geq d_1} T_{1i}(n) \ \Big(f_1(\vec{s}) = 1 \leftrightarrow f_0'' \prod_{i \in d_1} S_i(n) \times \prod_{i \geq d_1} \{s_i\} = \{1\} \Big).$$

Neither (15) nor (16) hold for function f_1, thus it is (17) which is applicable. Hence, a step analogous to the one we applied in case of function f_0 may be repeated with function f_1 as well. Generally speaking, the transfer for $n \in \omega$ from function f_n to function f_{n+1} is performed in a way which is analogous to the transfer from function f_0 to function f_1. In the final stage of the inductive procedure it becomes obvious that the assertion is right. □

Corollary. *Under the conditions of the Lemma* 5 *the following holds:*

$\exists k \in 2\, \exists \vec{t} \in \prod_{i \in \omega} T_i\ \exists \langle S_i \subseteq T_i : i \in \omega \rangle\ \exists A \in [\omega]^\omega\ \Big(\forall i \in \omega\, (S_i$ *is a downwards closed* ω*-subtree of the tree* T_i *with* $|\mathrm{IS}\,(t_i, T_i)|$ *branches which includes the set* $\mathrm{IS}\,(t_i, T_i)) \,\&\, A = \{|t_i| : i \in \omega\} \,\&\, \forall n \in \omega\, \forall j \in g(n)\, \exists i \in E_j\ (|t_i|$ *is the n-th member of the set* $A) \,\&\, f'' \bigotimes^{A}_{i \in \omega} S_i = \{k\} \Big)$. □

Now, we can apply the corollary to function h defined in (13). Without loss of generality, we can assume that $\Big| \bigotimes_{i \in \omega} \bigotimes_{j \in \omega} T''_{ij} \Big| = \omega$. Namely, it may be possible to choose $\vec{t} \in \prod_{i \in \omega} \prod_{j \in \omega} T''_{ij}$ so that $\Big| \bigotimes_{i \in \omega} \bigotimes_{j \in \omega} T''_{ij}[t_{ij}] \Big| = \omega$. Then we define set $E_i = \{\langle i, j \rangle : j \in \omega\}$ for each $i \in \omega$. In case we manage to define the increasing function g from ω into $\omega \setminus 1$ adequately, the corollary may be applied to function h, and then (12) and (9) which prove the statement. □

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Fruškogorska A–6,
25000 Sombor,
Yugoslavia

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