Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 6 (1995), 51–53.

A VOLUME INEQUALITY FOR A PARALLELEPIPED

Murray S. Klamkin

Dedicated to the memory of professor Dragoslav S. Mitrinović

We give a geometric proof of an inequality related to a parallelepiped.

Several years ago, according to G. D. Chakerian, S. Stein came across a review by Gauss of L. A. Seeber's "Untersuchungen über die Eigenshaften der positiven ternären quadratischen Formen", 1831. In this review Gauss observed that one of Seeber's theorems on quadratic forms could be interpreted geometrically as follows:

If a parallelepiped of edge lengts a, b, c is such that all of its face and body diagonals are longer than each of its edges, then its volume is larger than $abc/\sqrt{2}$.

Since this is an interesting geometric inequality with unusual constraints and since Seeber's paper is algebraic and unavailable, we give a geometric proof. Also, we make a small change and allow the face and body diagonals to be not less than the largest edge so that the volume can actually equal $abc/\sqrt{2}$.

We treat the problem via two cases (i) and (ii). In the first case, we do not need the body diagonal constraints. In the second, the body diagonal constraints are essential.

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ denote three concurrent vectors of respective lengths a, b, c, along three concurrent edges of the parallelepiped. Then its volume is given by

$$V = |\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Hence,

$$V^2 = \begin{vmatrix} \mathbf{A}\cdot\mathbf{A} & \mathbf{A}\cdot\mathbf{B} & \mathbf{A}\cdot\mathbf{C} \\ \mathbf{B}\cdot\mathbf{A} & \mathbf{B}\cdot\mathbf{B} & \mathbf{B}\cdot\mathbf{C} \\ \mathbf{C}\cdot\mathbf{A} & \mathbf{C}\cdot\mathbf{B} & \mathbf{C}\cdot\mathbf{C} \end{vmatrix} = (abc)^2 \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix},$$

[0]1991 Mathematics Subject Classification: 51M16

or

$$\left(\frac{V}{abc}\right)^2 = 1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma.$$

Here α, β, γ are the angles between pairs of the vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

The desired inequality is now

$$\frac{1}{2} \geq \cos^2\alpha + \cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma$$

and subject to the face diagonal length constraints. Since the face diagonals are given by $\mathbf{A}+\mathbf{B}$, $\mathbf{A}-\mathbf{B}$, etc., we must satisfy

$$(\mathbf{A}-\mathbf{B})^2 \geq \max(a^2, b^2, c^2), \quad (\mathbf{A}+\mathbf{B})^2 \geq \max(a^2, b^2, c^2), \text{ etc.},$$

or

$$\frac{1}{2} \geq \cos\alpha,\ \cos\beta,\ \cos\gamma \geq -\frac{1}{2}.$$

In the parallelepiped there exists at least one vertex such that the three face angles α, β, γ there are (i) all in $[0, \pi/2]$ or else (ii) are all in $[\pi/2, \pi]$. This is easy to check since all the faces are parallelograms.

Case (i). Here, $1/2 \geq \cos\alpha, \cos\beta, \cos\gamma \geq 0$ so that $90° \geq \alpha, \beta, \gamma \geq 60°$. Letting $x = \cos\alpha$, $y = \cos\beta$, $z = \cos\gamma$, the function

$$F \equiv \cos^2\alpha + \cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma = x^2 + y^2 + z^2 - 2xyz$$

is convex in each of the variables, x, y, z. Consequently, the maximum of F is taken on when x, y, z are either 0 or $1/2$. Thus, max $F = 1/2$ and is taken on for $\alpha = \beta = \gamma = \pi/3$ or $\alpha = \beta = \pi/3$, $\gamma = \pi/2$ (or permutations thereof).

Case (ii). This case is not so simple as the previous one.

Here on letting $x = -\cos\alpha$, $y = -\cos\beta$, $z = -\cos\gamma$, we want to show that

$$\frac{1}{2} \geq x^2 + y^2 + z^2 + 2xyz \equiv F,$$

where as before $1/2 \geq x, y, z \geq 0$. However, the upper bound $1/2$ is not good enough here. So assuming without loss of generality that $a \geq b \geq c$, we have by the face diagonal constraints that

$$0 \leq x \leq \frac{b^2 + c^2 - a^2}{2bc} \leq \frac{c}{2b}, \qquad 0 \leq y \leq \frac{c}{2b}, \qquad 0 \leq z \leq \frac{1}{2}.$$

Also by the body diagonal constraint, $(\mathbf{A}+\mathbf{B}+\mathbf{C})^2 \geq \mathbf{A}^2$ or

$$b^2 + c^2 \geq 2bcx + 2cay + 2abz \geq 2bcx + 2bcy + 2b^2z.$$

Hence,

$$x + y \leq \frac{b^2(1-2z) + c^2}{2bc} \equiv \lambda.$$

This leads to two sub cases: $(\mathrm{ii})_1$ the point $(x, y) = (c/(2b), c/(2b))$ lies below or on the line $x + y = \lambda$, $(\mathrm{ii})_2$ the point $(x, y) = (c/(2b), c/(2b))$ lies above the line $x + y = \lambda$.

Case $(\mathrm{ii})_1$. Here,

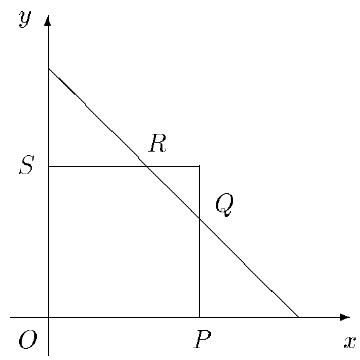
$$\frac{c}{b} \leq \frac{b^2(1-2z)+c^2}{2bc} \quad \text{or} \quad \frac{c^2}{b^2} \leq 1 - 2z.$$

Hence, $x, y \leq \sqrt{1-2z}\,/2$ and

$$x^2 + y^2 + z^2 + 2xyz \leq \frac{1-2z}{4} + \frac{1-2z}{4} + z^2 + \frac{z(1-2z)}{2} = \frac{1-z}{2} \leq \frac{1}{2}\,.$$

There is equality here if the face angles are 120°, 120°, and 90°.

Case $(\mathrm{ii})_2$. Here, $c^2/b^2 > 1 - 2z$ and (x, y) must lie in the convex pentagonal region $OPQRS$ as indicated in the figure. Now keeping z fixed, $F = x^2 + y^2 + z^2 + 2xyz$ is a convex function of (x, y) so that its maximum must occur at one of the vertices of $OPQRS$. By summetry, it suffices to just cosider points P and Q. At P, $y = 0$ and $x = c/(2b)$, so that



$$F = \frac{c^2}{4b^2} + z^2 \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}\,.$$

Again there is equality for face angles of 120°, 120°, and 90°.

At Q, $x = c/(2b)$ and $y = b(1-2z)/(2c)$, so that

$$F = \frac{c^2}{4b^2} + \frac{b^2(1-2z)^2}{4c^2} + z^2 + 2\left(\frac{c}{2b}\right)\left(\frac{b(1-2z)}{2c}\right)z.$$

Finally, since $1 \geq c^2/b^2 > 1 - 2z$,

$$F < \frac{1}{4} + \frac{1-2z}{4} + z^2 + \frac{(1-2z)}{2} = \frac{1}{2}\,.$$

Department of Mathematical Sciences,
University of Alberta,
Edmonton, Alberta,
Canada T6G 2G1

(Received July 20, 1995)