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TWO INEQUALITIES

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Dedicated to the memory of Professor Dragoslav S. Mitrinović

Inequalities (2) and (4) are proved.

An inequality that appear in MITRINOVIĆ [**1**, p. 247] is

$$\frac{3x}{2+\sqrt{1-x^2}} \le \sin^{-1} x, \quad 0 \le x \le 1. \tag{1}$$

This is best possible at $x=0$ but not at 1. It occurred to me to see if there is a corresponding upper bound for $\sin^{-1} x$. Indeed there is,

$$\sin^{-1} x \le \frac{\pi x}{2+\sqrt{1-x^2}}, \quad 0 \le x \le 1 \tag{2}$$

and equality holds at both ends of the interval. We give a proof of both inequalities. Let $f_a(x) = \dfrac{ax}{2+\sqrt{1-x^2}} = \sin^{-1} x$, $3 \le a \le \pi$, $0 \le x \le 1$. Note that $f_a(0)=0$ and $f_a(1) = \dfrac{a-\pi}{2} \le 0$ with $f_\pi(1)=0$. Then

$$f_a'(x) = \left(\sqrt{1-x^2}\left(2+\sqrt{1-x^2}\right)^2\right)^{-1} \left((2a-4)\sqrt{1-x^2} + a - 5 + x^2\right). \tag{3}$$

Call the second bracket $g_a(x) \equiv (2a-4)\sqrt{1-x^2} + a - 5 + x^2$ then $g_a'(a) = \dfrac{2x}{\sqrt{1-x^2}}\left(\sqrt{1-x^2} - (a-2)\right) \ne 0$ on $(0,1)$. So g is monotone decreasing. For $a = 3$, $g_3(0) = 0$ so $g_3(x) \le 0$ and $f_3'(2) \le 0$ as required to prove (1). For $a=\pi$, $g_\pi(x)$ has one zero and so $f_\pi(x)$ is unimodal with $f_\pi'(0) > 0$ and $f_\pi(0) = f_\pi(1) = 0$. This proves (2).

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Our second inequality concerns log convex functions. If $f > 0$ and $\log f$ is convex on $\mathbf{R}$ then

$$\frac{\pi}{4}\int_{-1}^{1} f(x+vt)\,\cos\,\frac{\pi t}{2}\,\mathrm{d}t \le \frac{f(x+v)+f(x-v)}{2}, \quad x, v \in \mathbf{R}. \tag{4}$$

The proof is to write

$$f(x+tv) \le f(x+v)^{\frac{1+t}{2}}\, f(x-v)^{\frac{1-t}{2}}, \quad -1 \le t \le 1.$$

Let $B = \dfrac{1}{2}\log\dfrac{f(x+v)}{f(x-v)}$, then

$$\begin{aligned}\frac{\pi}{4}\int_{-1}^{1} f(x+vt)\cos\frac{\pi t}{2}\,\mathrm{d}t &\le \big(f(x+v)f(x-v)\big)^{1/2}\,\frac{\pi}{4}\int_{-1}^{1}\cos\frac{\pi t}{2}\,e^{Bt}\,\mathrm{d}t\\ &= \frac{\pi^2}{8}\,\frac{e^B+e^{-B}}{B^2+\frac{\pi^2}{4}}\,\big(f(x+v)f(x-v)\big)^{1/2} = \frac{\frac{\pi^2}{4}}{B^2+\frac{\pi^2}{4}}\,\frac{f(x+v)f(x-v)}{2}.\end{aligned}$$

Equality holds in this argument if $f(x)$ is linear. In any case, (4) follows. Equality holds in (4) if f is a costant.

Moreover, if f is log concave we have

$$\frac{\pi}{4}\int_{-1}^{1} f(x+vt)\cos\frac{\pi t}{2}\,\mathrm{d}t \ge \frac{f(x+v)+f(x-v)}{2}\left(\frac{\frac{\pi^2}{4}}{\frac{\pi^2}{4}+\frac{1}{4}\log^2\frac{f(x+v)}{f(x-v)}}\right).$$

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