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A SHORT PROOF OF A FIXED POINT THEOREM IN NOT NECESSARILY LOCALLY CONVEX SPACES

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Dedicated to the memory of Professor Dragoslav S. Mitrinović

In this paper we give a new and simple proof of Hadžić's fixed point theorem [2].

In 1935. TYCHONOFF proved the following fixed point theorem:

Theorem A. TYCHONOFF. *If X is a Hausdorff locally convex topological vector space, and $K \subseteq X$ is its compact and convex nonempty subset, then every function $f : K \to K$ has at least one fixed point.*

In paper [**6**] K. ZIMA extended this theorem (in the case when X is metrizabile) on some classes of subsets, of paranormed (not nesessarily locally convex) spaces. Further extensions of TYCHONOFF's fixed point theorem are obtained by B. RZEPECKI [**5**], O. HADŽIĆ [**2**] (see also monograph [**3**]) and A. IDZIK [**4**].

In the paper [**1**] (p. 305, lemma 1.), KY FAN gave famous KKM principle which is an infinite dimensional generalization of the classical KKM (KNASTER–KURATOWSKI–MAZURKIEWICZ's) lemma, and using this result he also gave a new proof of TUCHONOFF's fixed point theorem. This proof is elementary, but it depends of properties of seminorms in locally convex topological vector space.

Let E be a vector space and $X \subseteq E$ a nonempty subset. A multifunction $F : X \to \mathcal{P}(E)$ is called KKM multifunction if $co\,(A) \subseteq \cup_{x\in A} F(x)$ for each finite subset $A \subseteq X$, where $co\,(.)$ denotes the convex hull.

Theorem B. (KY FAN). *Let E be a Hausdorff topological vector space, $X \subseteq E$ its arbitrary subset and $F : X \to \mathcal{P}(E)$ a KKM multifunction such that for some $x_0 \in X$, $F(x_0)$ is a compact and for each $x \in X$, $F(x)$ is closed in x. Then $\cap_{x\in X} F(x) \neq \emptyset$.*

In our paper we use this formulation of KKM principle, but our proof is simpler and shorter than FAN's proof of TYCHONOFF's fixed point theorem.

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Let E be a topological vector space, let $\mathcal{U}$ be the fundamental family of open neighbourhoods of zero in E and $K \subseteq E$. We say that the set K is of ZIMA's type if and only if for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ so that $co\,(U \cap (K-K)) \subseteq V$.

An example of such a subset in non locally convex space is given in [**3**].

Proposition. (O. HADŽIĆ [**2**]). *If X is a Hausdorff topological vector space, and $K \subseteq X$ is its compact and convex nonempty subset of Zima's type, then every continuous function $f : K \to K$ has at least one fixed point.*

Proof. Let $\mathcal{U}$ be the fundamental family of open neighborhoods of zero in X, and $f : K \to K$ a continuous function. It suffices to show that for each $V \in \mathcal{U}$ there exists a $x \in K$ such that $x - f(x) \in V$, because X is a compact HAUSDORFF space. Suppose that there exists a $V \in \mathcal{U}$ such that $x - f(x) \notin V$ for all $x \in K$. Since K is a set of ZIMA's type there exists $U \in \mathcal{U}$ so that $co\,\big(U \cap (K-K)\big) \subset V$. We define a multifunction $F : K \to \mathcal{P}(K)$ by:

$$F(x) = \{y \in K \mid \big(x - f(y)\big) \notin U\}.$$

Condition of theorem B, that $F(x)$ is closed, trivially holds for all $x \in K$. Suppose that F is not a KKM multifunction. Then there exists $x_0, \dots, x_n \in K$ such that $x_0 \in co\,(\{x_1, \dots, x_n\})$ but $x_0 \notin \cup_{k=1}^{n} F(x_k)$. Then $x_k - f(x_0) \in U$ for all $1 \le k \le n$. This implies, $x_0 - f(x_0) \in co\,\big(U \cap (K-K)\big) \subseteq V$, which is a contradiction. Hence, F is a KKM multifunction. By theorem B we have that there exists $y^* \in K$ such that $y^* \in F(x)$ for all $x \in K$. Then $y^* \in F\big(f(y^*)\big)$ i.e. $\big(f(y^*) - f(y^*)\big) = 0 \notin U$ which is a contradiction. Q.E.D.

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