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SOME NEW INEQUALITIES FOR GRAM DETERMINANTS IN INNER PRODUCT SPACES

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Dedicated to the memory of Professor Dragoslav S. Mitrinović

Some new inequalities are given for Gramians which complement the result from the recent book on the theory of inequalities due to Mitrinović, Pečarić and Fink.

1. INTRODUCTION

Let $(H;(\cdot,\cdot))$ be an inner product space over the real or complex number field $\mathbf{K}$ and $\{x_1,\dots,x_n\}$ a system of vectors in H. Consider the GRAM matrix $G(x_1,\dots,x_n) := [(x_i,x_j)]_{i,j=\overline{1,n}}$ and the Gram determinant $\Gamma(x_1,\dots,x_n) := \det G(x_1,\dots,x_n)$.

The following inequality is well known in the literature as GRAM's inequality (see [**4**, p. 395])

$$\Gamma(x_1,\dots,x_n) \geq 0. \tag{1.1}$$

The equality holds in (1.1) iff the system of vectors $\{x_1,\dots,x_n\}$ is linearly dependent in H.

A well known converse of this inequality is the following:

$$\Gamma(x_1,\dots,x_n) \leq \prod_{i=1}^{n} \|x_i\|^2 \quad \text{for all } x_i \in H \quad (i=\overline{1,n}) \tag{1.2}$$

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known as HADAMARD's inequality. Note that the equality in (1.2) holds if and only if $(x_i, x_j) = \delta_{ij}\|x_i\|\,\|x_j\|$ for all $i, j \in \{1, \ldots, n\}$ (see [**3**] and [**4**], p. 597]).

Other special inequalities which involve GRAM's determinant are [**4**, p. 597]:

$$\frac{\Gamma(x_1, \ldots, x_n)}{\Gamma(x_1, \ldots, x_k)} \le \frac{\Gamma(x_2, \ldots, x_n)}{\Gamma(x_2, \ldots, x_k)} \le \ldots \le \Gamma(x_{k+1}, \ldots, x_n), \tag{1.3}$$

$$\Gamma(x_1, \ldots, x_n) \le \Gamma(x_1, \ldots, x_k)\,\Gamma(x_{k+1}, \ldots, x_n) \tag{1.4}$$

and

$$\Gamma(x_1 + y_1, x_2, \ldots, x_n)^{1/2} \le \Gamma(x_1, x_2, \ldots, x_n)^{1/2} + \Gamma(y_1, x_2, \ldots, x_n)^{1/2}. \tag{1.5}$$

In what follows we use the next known lemma.

Lemma 1.1. *Let $(H; (\cdot, \cdot))$ be an inner product space and $\{e_1, \ldots, e_n\}$ a system of linearly independent vectors in H. Put $H_n := Sp\{e_1, \ldots, e_n\}$ the linear space spaned by $\{e_1, \ldots, e_n\}$. Then we have the representation*:

$$d^2(u, H_n) = \frac{\Gamma(u, e_1, \ldots, e_n)}{\Gamma(e_1, \ldots, e_n)}. \tag{1.6}$$

In this paper we point out some new inequalities for Gramians closely connected with those in (1.2)–(1.4). Note that these results complement in a natural way Chapter XX of the recent book due to D. S. MITRINOVIĆ, J. E. PEČARIĆ and A. M. FINK [**4**].

2. RESULTS

Let $(H; (\cdot, \cdot))$ be an inner product space over the real or complex number field $\mathbf{K}$. The following inequality is well known in the literature as SCHWARZ's inequality

$$\|x\|^2\|y\|^2 \ge |(x, y)|^2 \text{ for all } x, y \in H. \tag{2.1}$$

Note that the equality holds in (2.1) if and only if there exists a constant $\lambda \in \mathbf{K}$ so that $x = \lambda y$ or $y = 0$ or $x = 0$.

The following interesting refinement of (2.1) in terms of Gramians holds.

Theorem 2.1. *Let $(H(\cdot, \cdot))$ be as above and $\{x_1, \ldots, x_n\}$ a system of linearly independent vectors in H. Denote by H_n the linear subspace spaned by $\{x_1, \ldots, x_n\}$. Then for all $x \in H$ and $y \in H_n$ one has the inequality*:

$$\|x\|^2\|y\|^2 \ge |(x, y)|^2 + \|y\|^2\, \frac{\Gamma(x, x_1, \ldots, x_n)}{\Gamma(x_1, \ldots, x_n)} \ge |(x, y)|^2. \tag{2.2}$$

Proof. By Lemma 1.1 we have:

$$\frac{\Gamma(x, x_1, \ldots, x_n)}{\Gamma(x_1, \ldots, x_n)} = d^2(x, H_n) = \inf_{z \in H_n} \|x - z\|^2 \le \|x - \lambda y\|^2$$

for all $\lambda \in \mathbf{K}$, from which we get:

$$\frac{\Gamma(x, x_1, \dots, x_n)}{\Gamma(x_1, \dots, x_n)} \le \|x\|^2 - 2\,\mathrm{Re}\,[\overline{\lambda}(x, y)] + |\lambda|^2 \|y\|^2$$

for all $\lambda \in \mathbf{K}$.

If we choose $\lambda = \dfrac{(x, y)}{\|y\|^2} \in \mathbf{K}$, $(y \ne 0)$ then we get in the second part of the above inequality:

$$\|x\|^2 - 2\,\frac{|(x, y)|^2}{\|y\|^2} + \frac{|(x, y)|^2}{\|y\|^2} = \frac{\|x\|^2\|y\|^2 - |(x, y)|^2}{\|y\|^2},$$

which proves the first part of (2.2).

The second part is obvious.

If $y = 0$, (2.2) becomes an equality.

Remark 2.2. Note that the equality holds simultaneously in (2.2) iff $x = \lambda y$ with $\lambda \in \mathbf{K}$ or $y = 0$ or $x = 0$.

The following corollary also holds:

Corollary 2.3. *Let $(H(\cdot, \cdot))$ be an inner product space and $\{x_1, \dots, x_n\}$ be as above. Then for all $x, z \in H$ and $y, u \in H_n$, one has the equality*

$$\|x\|\,\|z\|\,\|y\|\,\|u\| - |(x, y)(z, u)| \ge \|y\|\,\|u\|\,\frac{|\gamma(x_1, \dots, x_n)(u, z)|}{\Gamma(x_1, \dots, x_n)} \ge 0, \tag{2.3}$$

where

$$\gamma(x_1, \dots, x_n)(x, z) := \begin{bmatrix} (x, z) & (x, x_1) & \cdots & (x, x_n) \\ (x_1, z) & & & \\ \cdots & & G(x_1, \dots, x_n) & \\ (x_n, z) & & & \end{bmatrix}.$$

Proof. By the above theorem we have:

$$\|x\|^2\|y\|^2 - |(x, y)|^2 \ge \|y\|^2\,\frac{\Gamma(x, x_1, \dots, x_n)}{\Gamma(x_1, \dots, x_n)}$$

and

$$\|z\|^2\|u\|^2 - |(z, u)|^2 \ge \|u\|^2\,\frac{\Gamma(z, x_1, \dots, x_n)}{\Gamma(x_1, \dots, x_n)}$$

from which we get

$$\begin{aligned} &\left(\|x\|^2\|y\|^2 - |(x, y)|^2\right)\left(\|z\|^2\|u\|^2 - |(z, u)|^2\right) \\ &\qquad \ge \|y\|^2\|u\|^2\,\frac{\Gamma(x, x_1, \dots, x_n)\,\Gamma(z, x_1, \dots, x_n)}{[\Gamma(x_1, \dots, x_n)]^2}. \end{aligned} \tag{2.4}$$

A simple calculations shows us that:

$$\left(\|x\|\,\|y\|\,\|z\|\,\|u\|-|(x,y)|\,|(z,u)|\right)^2 \geq \left(\|x\|^2\|y\|^2-|(x,y)|^2\right)\left(\|z\|^2\|u\|^2-|(z,u)|^2\right). \tag{2.5}$$

It is known that (see [**2**] or the inequality of Kurepa [**4**, p. 599] in a particular case):

$$\Gamma(x,x_1,\ldots,x_n)\,\Gamma(z,x_1,\ldots,x_n) \geq |\gamma(x_1,\ldots,x_n)(x,z)|^2. \tag{2.6}$$

Now, combining the inequalities (2.4), (2.5) and (2.6) we easily derive (2.3).

Remark 2.4. If we choose in the above inequality $x = z$ and $y = u$, we recover the first inequality in (2.2).

The following general refinement of Schwarz's inequality in terms of Gramians also holds:

Theorem 2.5. *Let $(H;(\cdot,\cdot))$ be an inner product space and $\{x_1,\ldots,x_n\}$ a family of linearly independent vectors in H. Thus for all $x,y \in H\setminus\{0\}$ one has the inequality:*

$$\|x\|^2\|y\|^2-|(x,y)|^2 \geq \frac{\max\{I(x),I(y)\}}{\Gamma(x_1,\ldots,x_n)} \geq 0, \tag{2.7}$$

where

$$I(x)=\frac{1}{\|x\|^2}\left(\|x\|^2[\Gamma(y,x_1,\ldots,x_n)]^{1/2}-|(x,y)|\,[\Gamma(x,x_1,\ldots,x_n)]^{1/2}\right)^2$$

and

$$I(y)=\frac{1}{\|y\|^2}\left(\|y\|^2[\Gamma(x,x_1,\ldots,x_n)]^{1/2}-|(x,y)|\,[\Gamma(y,x_1,\ldots,x_n)]^{1/2}\right)^2.$$

Proof. By the known inequality (1.5) we have:

$$\left|[\Gamma(a,x_1,\ldots,x_n)]^{1/2} - [\Gamma(b,x_1,\ldots,x_n)]^{1/2}\right| \leq [\Gamma(a-b,x_1,\ldots,x_n)]^{1/2} \leq \|a-b\|\,[\Gamma(x_1,\ldots,x_n)]^{1/2}$$

for all $a,b \in H$ which gives us:

$$\left([\Gamma(a,x_1,\ldots,x_n)]^{1/2}-[\Gamma(b,x_1,\ldots,x_n)]^{1/2}\right)^2 \leq \|a-b\|^2\,\Gamma(x_1,\ldots,x_n) \tag{2.8}$$

for all $(a,b) \in H$.

Choose $a = x$ and $b = \dfrac{(x,y)}{\|y\|^2}\,y$. Then a simple calculations shows that:

$$\|a-b\|^2=\frac{\|x\|^2\|y\|^2-|(x,y)|^2}{\|y\|^2}$$

and

$$\left([\Gamma(a,x_1,\ldots,x_n)]^{1/2}-[\Gamma(b,x_1,\ldots,x_n)]^{1/2}\right)^2$$
$$=\left([\Gamma(x,x_1,\ldots,x_n)]^{1/2}-\frac{|(x,y)|}{\|y\|^2}[\Gamma(y,x_1,\ldots,x_n)]^{1/2}\right)^2.$$

Now, using the inequality (2.8), we get (2.7) for the mapping $I(y)$. By a similar argument we can show the second inequality, and the proof of the Theorem is thus finished.

The following inequality which generalizes the well known triangle inequality in inner product spaces (see for example [**4**, p. 598]) holds:

$$[\Gamma(x+y,x_1,\ldots,x_n)]^{1/2}\le[\Gamma(x,x_1,\ldots,x_n)]^{1/2}+[\Gamma(y,x_1,\ldots,x_n)]^{1/2},$$

where $x_i\in H$ $(i=\overline{1,n})$ and $(x,y)\in H$.

We are interested here in a converse of this interesting result.

Theorem 2.6. *Let* $(H;(\cdot,\cdot))$ *and* $\{x_1,\ldots,x_n\}$ *be as above. Then for all* $x,y\in H$ *one has the inequality:*

$$\Gamma(x+y,x_1,\ldots,x_n)+\|x-y\|^2\,\Gamma(x_1,\ldots,x_n)\ge 2\,[\Gamma(x,x_1,\ldots,x_n)+\Gamma(y,x_1,\ldots,x_n)]. \tag{2.9}$$

Proof. If $\{x_1,\ldots,x_n\}$ is linearly dependent the inequality is obvious.

Suppose that $\{x_1,\ldots,x_n\}$ is linearly independent. Then by Lemma 1.1 we have:

$$\frac{\Gamma(x+y,x_1,\ldots,x_n)}{\Gamma(x,x_1,\ldots,x_n)}=d^2(x+y,H_n)=\inf_{z\in H_n}\|x+y-z\|^2=\inf_{u\in H_n}\|x+y-2u\|^2.$$

Using the parallelogram identity:

$$\|x+y-2u\|^2+\|x-y\|^2=2\left(\|x-u\|^2+\|y-u\|^2\right)\ \text{ for all } x,y,u\in H$$

we get:

$$\begin{aligned}\inf_{u\in H_n}\|x+y-2u\|^2 &= \inf_{u\in H_n}\left\{2\left(\|x-u\|^2+\|y-u\|^2\right)-\|x-y\|^2\right\}\\ &\ge 2\Big[\inf_{u\in H_n}\|x-u\|^2+\inf_{u\in H_n}\|y-u\|^2\Big]-\|x-y\|^2,\end{aligned}$$

i.e.

$$\frac{\Gamma(x+y,x_1,\ldots,x_n)}{\Gamma(x_1,\ldots,x_n)}+\|x-y\|^2\ge 2\left[\frac{\Gamma(x,x_1,\ldots,x_n)}{\Gamma(x_1,\ldots,x_n)}+\frac{\Gamma(y,x_1,\ldots,x_n)}{\Gamma(x_1,\ldots,x_n)}\right]$$

and the theorem is proved.

REMARK 2.7. If we change y with $(-y)$ in (2.9), we get the similar inequality:

$$\Gamma(x-y,x_1,\ldots,x_n)+\|x+y\|^2\,\Gamma(x_1,\ldots,x_n) \geq 2\,[\Gamma(x,x_1,\ldots,x_n)+\Gamma(y,x_1,\ldots,x_n)], \tag{2.10}$$

where x,y and x_i $(i=\overline{1,n})$ are as above.

Thus, we can state that:

$$\begin{aligned} &\min\Big\{\Gamma(x-y,x_1,\ldots,x_n)+\|x+y\|^2\,\Gamma(x_1,\ldots,x_n),\\ &\Gamma(x+y,x_1,\ldots,x_n)+\|x-y\|^2\,\Gamma(x_1,\ldots,x_n)\Big\}\\ &\geq 2\big[\Gamma(x,x_1,\ldots,x_n)+\Gamma(y,x_1,\ldots,x_n)\big]. \end{aligned} \tag{2.11}$$

Furthermore, by the elementary inequality

$$2(a^2+b^2)\geq(a+b)^2 \text{ for all } a,b\in\mathbf{R},$$

we get

$$\begin{aligned} 2\big[\Gamma(x,x_1,\ldots,x_n) \ &+\ \Gamma(y,x_1,\ldots,x_n)\big]\\ &\geq \Big([\Gamma(x,x_1,\ldots,x_n)]^{1/2}+[\Gamma(y,x_1,\ldots,x_n)]^{1/2}\Big)^2 \end{aligned}$$

which gives, by inequality (2.11),

$$\begin{aligned} &[\Gamma(x,x_1,\ldots,x_n)]^{1/2}+[\Gamma(y,x_1,\ldots,x_n)]^{1/2}\\ &\leq \min\Big\{\big(\Gamma(x-y,x_1,\ldots,x_n)+\|x+y\|^2\,\Gamma(x_1,\ldots,x_n)\big)^{1/2},\\ &\big(\Gamma(x+y,x_1,\ldots,x_n)+\|x-y\|^2\,\Gamma(x_1,\ldots,x_n)\big)^{1/2}\Big\} \end{aligned} \tag{2.12}$$

i.e., converse of (1.5).

Now, we can give another results for Gramians.

Theorem 2.8. *Let $(H;(\cdot,\cdot))$ be an inner product space, $\{x_1,\ldots,x_n\}$ a system of linearly independent vectors in H and $k\in\{2,\ldots,n-1\}$, $(n\geq 3)$. Then one has the inequality:*

$$\frac{\Gamma\left(\frac{x+y}{2},x_1,\ldots,x_n\right)}{\Gamma(x_1,\ldots,x_n)}\leq\frac{1}{2}\left[\frac{\Gamma(x,x_1,\ldots,x_k)}{\Gamma(x_1,\ldots,x_k)}+\frac{\Gamma(y,x_{k+1},\ldots,x_n)}{\Gamma(x_{k+1},\ldots,x_n)}\right] \tag{2.13}$$

for all $x,y\in H$.

Proof. Fix $k\in\{2,\ldots,n-1\}$ and denote $E:=Sp\{x_1,\ldots,x_k\}$ and $E_2:=Sp\{x_{k+1},\ldots,x_n\}$ and $E:=E_1\oplus E_2$, i.e., the direct sum of the linear subspaces E_1 and E_2, this means that, for every $x\in E$, there exists a unique $x_1\in E_1$ and a unique $x_2\in E_2$ so that $x=x_1+x_2$.

By the parallelogram identity one has:

$$\|x+y-(y_1+y_2)\|^2+\|x-y-y_1+y_2\|^2=2\left(\|x-y_1\|^2+\|y-y_2\|^2\right)$$

for all $x,y\in H$ and $(y_1,y_2)\in E_1\times E_2$, which give us

$$\|x+y-(y_1+y_2)\|^2\le 2\left(\|x-y_1\|^2+\|y-y_2\|^2\right)$$

for all $x,y\in H$ and $(y_1,y_2)\in E_1\times E_2$.

Now, we have:

$$\begin{aligned}&\inf_{(y_1,y_2)\in E_1\times E_2}\|x+y-(y_1+y_2)\|^2\\ &\le \inf_{(y_1,y_2)\in E_1\times E_2}\left[2\left(\|x-y_1\|^2+\|y-y_2\|^2\right)\right]\\ &=2\left[d^2(x,E_1)+d^2(e,E_2)\right].\end{aligned}\tag{2.14}$$

On the other hand we have

$$\inf_{(y_1,y_2)\in E_1\times E_2}\|x+y-(y_1+y_2)\|^2=\inf_{z\in E}\|x+y-z\|^2=d^2(x+y,E)$$

which gives, by (2.11), that:

$$d^2(x+y,E)\le 2\left[d^2(x,E_1)+d^2(x,E_2)\right],\tag{2.15}$$

since

$$d^2(x,E_1)=\frac{\Gamma(x,x_1,\dots,x_k)}{\Gamma(x_1,\dots,x_k)},\quad d^2(x,E_2)=\frac{\Gamma(x,x_{k+1},\dots,x_n)}{\Gamma(x_{k+1},\dots,x_n)}$$

and

$$d^2(x,E)=\frac{\Gamma(x,x_1,\dots,x_n)}{\Gamma(x_1,\dots,x_n)}$$

the inequality (2.15) gives us the desired result (2.13).

Remark 2.9. The above theorem gives us the possibility to consider the following functional:

$$\psi(x,I,S):=\frac{\Gamma(x,x_{i_1},\dots,x_{i_n})}{\Gamma(x_{i_1},\dots,x_{i_n})},$$

where $x\in H$, I is a finite part of $\mathcal{A}$ with $I=\{i_1,\dots,i_n\}$ and $S=\{x_i\}_{i\in\mathcal{A}}$ is a linearly independent system of vectors in inner product space $(H;(\cdot,\cdot))$.

By a similar argument as in the above theorem we can prove that

$$\psi\left(\frac{x+y}{2},I\cup J,S\right)\le\left[\psi(x,I,S)+\psi(y,J,S)\right]\tag{2.16}$$

for all $x,y\in H$ and I,J disjoint finite parts of $\mathcal{A}$.

The following corollary of theorem 2.8 also holds:

Corollary 2.10. *With the above assumptions we have*:

$$\frac{\Gamma(x, x_1, \ldots, x_n)}{\Gamma(x_1, \ldots, x_n)} \le \frac{1}{2}\left[\frac{\Gamma(x, x_1, \ldots, x_k)}{\Gamma(x_1, \ldots, x_k)} + \frac{\Gamma(x, x_{k+1}, \ldots, x_n)}{\Gamma(x_{k+1}, \ldots, x_n)}\right] \tag{2.17}$$

for all $x \in H$.

For other recent results connected with the Gram determinant see the papers [**1**], [**2**] and the book [**4**, Chapter XX] where further references are given.

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