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CONNECTEDNESS OF THE GENERALIZED DIRECT PRODUCT OF DIGRAPHS

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Dedicated to the memory of Professor Dragoslav S. Mitrinović

Using spectral techniques we prove a theorem giving a necessary and sufficient condition for a generalized direct product (GDP) of strongly connected digraphs (with some additional restrictions) to be strongly connected. In the case of disconnectedness the number of strong components is given.

The necessary background and terminology can be found in [**2**]. We will limit ourselves to defining only lesser known terms and those which may cause confusion.

By a digraph throughout this paper we mean a digraph in which both loops and multiple arcs are allowed. More precisely, a digraph is an ordered pair $G = (V, E)$, where $V(G) = V$ is a finite non-empty set and $E(G) = E$ is a family of ordered pairs of V (multiplicity of which can of course exceed 1). A (undirected) graph is a symmetric digraph. A digraph G is called *complete* if each ordered pair of vertices u, v of G (if loops are not allowed then $u \neq v$) belongs to $E(G)$ with the same multiplicity. A digraph is *regular* of degree r if each indegree and each outdegree is equal to r. The *cycle* (directed) is a (strongly) connected regular digraph of degree 1. A strongly connected digraph G is called *bipartite* if it has no odd cycles, or equivalently if the vertex set V of G can be partitioned into two subsets V_1 and V_2 such that every arc of G joins a vertex of V_i to a vertex of V_j, $i \neq j$. A bipartite digraph G (with partite sets V_1 and V_2) having additional property that each ordered pair (u, v), $(u \in V_i,\ v \in V_j,\ i \neq j)$, belongs to $E(G)$, with the same multiplicity, is called *bicomplete*. By $\vec{C}_p$ (C_p) we denote the directed (undirected) cycle with p vertices, all arcs of which have the same multiplicity.

The spectrum of a digraph G is the spectrum of its adjacency matrix $A(G) = [a_{ij}]_1^p$, where $|V(G)| = p$ and $a_{ij} \geq 0$ is the number of arcs leading from the vertex corresponding to ith row to the vertex corresponding to jth column of A. The *index*

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r of a strongly connected digraph G is its the greatest real eigenvalue. As is known (theorem of FROBENIUS) $|\lambda_i| \leq r$ holds, for all eigenvalues λ_i of G.

Let G be a digraph with at most ν parallel arcs between any two vertices or loops of a vertex in G (if there are no parallel arcs then $\nu = 1$), then *complement* $\overline{G}$ of G is the digraph which has the same set of vertices as G and for any ordered pair (u, v) of vertices u and v of $\overline{G}$ (if loops are not allowed then $u \neq v$) from u to v lead $\nu - a$ arcs, where a is the number of arcs leading from u to v in G.

Definition 1. *Let $B \subseteq \{1, 0, -1\}^n \setminus \{(0, 0, \ldots, 0)\}$. The generalized direct product with basis B of digraphs $G_1, G_2, \ldots, G_n$ is the digraph $G = GDP(B; G_1, G_2, \ldots, G_n)$ whose vertex set is the Cartesian product of the vertex sets of digraphs $G_1, G_2, \ldots, G_n$. For two vertices say $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ of G construct all the possible arc selections of the following type. For each $n-$tuple $(\beta_1, \beta_2, \ldots, \beta_n) \in B$, such that $u_k = v_k$ holds whenever $\beta_k = 0$, select an arc going from u_i to v_i in G_i whenever $\beta_i = 1$ and an arc going from u_i to v_i in $\overline{G_i}$ whenever $\beta_i = -1$. The number of arcs going from u to v in G is equal to the number of such selections.*

If B consists of $n-$tuples of symbols 1 and 0, only, the resulting operation is called the non-complete extended $p-$sum (NEPS). The $p-$sum is obtained if B consists of all the possible $n-$tuples with exactly p $1's$. If $p = n$, the $p-$sum is called the product. The $1-$sum is also called the sum. The NEPS, basis of which contains all the possible $n-$tuples, is called the strong product.

We shall investigate connectedness of the generalized direct product (GDP) of strongly connected digraphs by using Theorems 0.4 and 0.5 from [**2**].

For that purpose we need the following results from [**4**] and [**5**].

Theorem 1. *Let G be a regular digraph with p vertices, degree r, and maximum number of parallel arcs between any two vertices or loops of a vertex equal to ν and let $\lambda_1 = r, \lambda_2, \cdots, \lambda_p$ be the spectrum of G. The complement $\overline{G}$ of G has the spectrum given by: $\overline{\lambda}_1 = \nu p - \nu - r$, $\overline{\lambda}_2 = -\nu - \lambda_2, \ldots, \overline{\lambda}_p = -\nu - \lambda_p$, if loops are not allowed, and $\overline{\lambda}_1 = \nu \cdot p - r, \overline{\lambda}_2 = -\lambda_2, \ldots, \overline{\lambda}_p = -\lambda_p$, if loops are allowed in G ($\overline{G}$).*

The eigenvectors belonging to λ_i and $\overline{\lambda}_i$ are the same and the eigenvector belonging to the eigenvalue λ distinct from r is orthogonal to the eigenvector $(1, 1, \ldots, 1)$ belonging to r.

Let A_i ($\overline{A}_i$) denote the adjacency matrix of (complement $\overline{G}_i$ of) the digraph G_i, $A_i^{[1]} = A_i$, $A_i^{[0]} = I$ (of the same order), $A_i^{[-1]} = \overline{A}_i$ and let $\otimes$ denote the KRONECKER product of matrices.

Theorem 2. *The generalized direct product with basis B of digraphs $G_1, G_2, \ldots, G_n$ has the adjacency matrix A given by*

$$A = \sum_{\beta \in B} A_1^{[\beta_1]} \otimes A_2^{[\beta_2]} \otimes \cdots \otimes A_n^{[\beta_n]}.$$

The following theorem is a slight generalization of Theorem 5 from [**4**]. Its proof coincides with the proof of Theorem 2.23 in [**2**].

Theorem 3. *If, for $i = 1, 2, \ldots, n$, λ_{ij_i}, $(\overline{\lambda}_{ij_i})$, $j_i = 1, 2, \ldots, p_i$, is the spectrum of a digraph G_i (complement $\overline{G}_i$ of G_i, given by Theorem 1 in the case of regularity of G_i) (p_i being its number of vertices), then the spectrum of GDP$(B; G_1, G_2, \ldots, G_n)$, in which G_i is a regular digraph whenever there exists $\beta \in B$ such that $\beta_i = -1$, consists of all possible values $\Lambda_{j_1, j_2, \cdots, j_n}$ where*

$$\Lambda_{j_1, j_2, \cdots, j_n} = \sum_{\beta \in B} \lambda_{1j_1}^{[\beta_1]} \cdot \lambda_{2j_2}^{[\beta_2]} \cdots \lambda_{nj_n}^{[\beta_n]},$$

$$\lambda_{ij_i}^{[1]} = \lambda_{ij_i},\ \lambda_{ij_i}^{[0]} = 1,\ \lambda_{ij_i}^{[-1]} = \overline{\lambda}_{ij_i},\ j_i = 1, 2, \ldots, p_i;\ i = 1, 2, \cdots, n.$$

The eigenvector $x_{j_1, j_2, \ldots, j_n} = x_{1j_1} \otimes x_{2j_2} \otimes \cdots \otimes x_{nj_n}$ belongs to the eigenvalue $\Lambda_{j_1, j_2, \ldots, j_n}$, where x_{ij_i} is an eigenvector belonging to the eigenvalue λ_{ij_i} of G_i.

We shall consider the GDP, basis B of which has property (D): for each $j \in \{1, 2, \ldots, n\}$ the set $\{\beta_j \,|\, \beta \in B\}$ is not a subset of $\{0, -1\}$. This condition implies that the GDP, effectively, depends on each G_i. However, this condition does not represent an essential restriction in investigation of connectedness of a GDP, because in the case $\{\beta_j \,|\, \beta \in B\} \subseteq \{0, -1\}$ for some j, we can replace G_j by its complement $\overline{G}_j$, provided in each $n-$tuple $\beta \in B$ the $j-$th coordinate -1 is replaced by 1 [**4**], while the case when all β_j are equal to 0 is not interesting and will be excluded from consideration.

Let $h(G) = h$ be the greatest common divisor of the lengths of all the cycles in a digraph G. The digraph G is called *primitive* if it is strongly connected and $h = 1$ and *imprimitive* if it is strongly connected and $h > 1$. In the second case h is called *the index of imprimitivity* (h is the index of imprimitivity of the adjacency matrix of the digraph G as well ([**1**], p. 183)).

The following simple assertion [**6**] will be used many times in the sequel.

Lemma 1. *If a regular, connected, imprimitive digraph, without multiple arcs, has even number of vertices and degree equal to half of the number of vertices, then this digraph is bicomplete.*

Proof. Since the index of imprimitivity of a digraph divides all lengths of its cycles it is only necessary to show that this digraph has a cycle of length two. If r is the degree of this digraph then it has $2r^2$ arcs but in the case of absence of symmetric arcs the maximum possible number of arcs is $\binom{2r}{2} = r(2r-1)$. Thus, the statement follows.

Similar assertion holds also for digraphs containing multiple arcs.

It is noticed in [**4**] that a GDP of regular digraphs is a regular digraph and that (weak) components of a regular digraph are its strong components too.

A *maximal eigenvalue* of a digraph G is an eigenvalue of G, modulus of which is equal to the index of G. In investigating the connectedness of GDP, using spectral techniques, we give rise naturally to the following question.

Question. *Which connected, regular, imprimitive digraphs G have property* (M)*: There exists a maximal eigenvalue of G, different from the index, such that by Theorem* 1 *corresponding eigenvalue of $\overline{G}$ is maximal too ?*

The answer to this question is given by the following lemma.

Lemma 2. *The property* (M) *have only regular, bicomplete digraphs and, in the case when loops are not allowed, regular, bicomplete digraphs and the cycle of length 3, with the same multiplicity of all arcs* (*in both cases*).

In the case of bicomplete digraphs the argument of the corresponding eigenvalue of $\overline{G}$ *is equal to zero and in the case of the cycle of length 3 it is twice greater than the argument of the corresponding eigenvalue of* G.

Proof. The index $\nu p - r - \nu l(G)$ of the complement $\overline{G}$, of a regular digraph G is corresponded to the index r of G, where ν is the maximal number of parallel arcs between any two vertices or loops of a vertex in G and $l(G) = 1$ if loops are forbidden, and $l(G) = 0$ otherwise.

Let h be the index of imprimitivity of G. All eigenvalues of G, which have modulus equal to r, can be written in the form $re^{i\frac{\ell}{h}2\pi}$, $\ell = 0, 1, \ldots, h-1$ (by a well-known theorem of Frobenius). The question is: when does to any eigenvalue $re^{i\frac{\ell}{h}2\pi}$, $1 \le \ell \le h-1$, of a regular digraph G, correspond eigenvalue $(\nu p - r - \nu l(G))e^{i\Theta}$ of $\overline{G}$, for any Θ. According to Theorem 1, the eigenvalue

$$-\nu l(G) - re^{i\frac{\ell}{h}2\pi} = \left(r^2 + \left(2\nu r \cos\frac{\ell}{h}2\pi + \nu^2\right) l(G)\right)^{\frac{1}{2}} e^{i\Theta},$$

of $\overline{G}$, where $\Theta = \arg\left(-\nu \cdot l(G) - re^{i\frac{\ell}{h}2\pi}\right)$, is corresponded to the eigenvalue $re^{i\frac{\ell}{h}2\pi}$ of G.

Consider, firstly, the case when loops are forbidden. The following two cases are occur: 1° $\nu p - r - \nu = \left(r^2 + 2\nu r \cos\frac{\ell}{h}2\pi + \nu^2\right)^{\frac{1}{2}} = r - \nu$ and 2° $\nu p - r - \nu = \left(r^2 + 2\nu r \cos\frac{\ell}{h}2\pi + \nu^2\right)^{\frac{1}{2}} = r$. The case $\left(r^2 + 2\nu r \cos\frac{\ell}{h}2\pi + \nu^2\right)^{\frac{1}{2}} = r + \nu$ is impossible, since $\ell > 0$. It is easy to check that the cases $(r^2 + 2\nu r \cos\frac{\ell}{h}2\pi + \nu^2)^{\frac{1}{2}} = r + k$, $k \in Z$, $0 < |k| < \nu$, are also impossible. In order to see this note that $\overline{G}$ is strongly connected and primitive if $h(G) \ge 4$.

In the case 1°, $\cos\frac{\ell}{h}2\pi = -1$ holds, which gives $h = 2\ell$, $2r = \nu p$ and according to Lemma 1, G is bicomplete, each arc of which has the same multiplicity – equal to ν. In this case $\Theta = 0$.

In the case 2°, $\cos\frac{\ell}{h}2\pi = -\frac{\nu}{2r}$ holds, which gives (due to the rationality of the cosine of angles of the form $\frac{\ell}{h}2\pi$ and $r \ge \nu$) $r = \nu$ and since $2r = \nu p - \nu$ we have $p = 3$ and consequently G is isomorphic to $\vec{C}_3$, each arc of which has multiplicity ν. In this case $\Theta = \frac{4\pi}{3}$ if $\ell = 1$ and $\Theta = \frac{2\pi}{3}$ if $\ell = 2$.

If loops are allowed in G, $\overline{G}$, the modulus of the corresponding eigenvalues of G and $\overline{G}$ (different from the indices) are the same, and arguments differ by π, and from $r = \nu p - r$, by Lemma 1, it follows that G is bicomplete, each arc of which has multiplicity ν, and $\Theta = \frac{\ell}{h}2\pi + \pi = 0 \,(\mathrm{mod}\, 2\pi)$.

This completes the proof of the lemma.

Definition 2. *A subset* $\{i_1, i_2, \ldots, i_s\}$ *of* $\{1, 2, \ldots, n\}$ *is consistent with digraphs* $G_1, G_2, \ldots, G_n$ *with respect to the basis* B *if for each* $k \in \{i_1, i_2, \ldots, i_s\} \cap \{\nu | \exists \beta \in$

$B \wedge \beta_\nu = -1\}$, *the digraph* G_k *is bicomplete or, in the case when loops are forbidden, bicomplete or isomorphic to* $\vec{C}_3$.

From our Theorem 3 and Theorem 0.5 in [**2**] the following assertion follows.

Theorem 4. *If* $G_1, G_2, \dots, G_n$ *are strongly connected digraphs then, under conditions of Theorem* 3, *the* (*weak*) *components of* $GDP(B; G_1, \dots, G_n)$ *are its strong components too.*

Definition 3. *If integers* $x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}, x^{(0)}$ *satisfy the equation*

$$\frac{x_1}{h_1} + \frac{x_2}{h_2} + \cdots + \frac{x_n}{h_n} = x \tag{1}$$

on $n+1$ *variables in integers* $x_1, x_2, \cdots, x_n, x$ *where* $h_1, h_2, \cdots, h_n$, $(n > 0)$ *are natural numbers, then the classes* $x_1 = x_1^{(0)} (\operatorname{mod} h_1), x_2 = x_2^{(0)} (\operatorname{mod} h_2), \dots, x_n = x_n^{(0)} (\operatorname{mod} h_n)$ *are called a solution of equation* (1).

It can be shown that equation (1) has exactly[1] $\dfrac{h_1 \cdot h_2 \cdots h_n}{\text{l.c.m.}\,(h_1, h_2, \dots, h_n)}$ solutions.

For a digraph G let $e(G)$ be defined as follows:

$$e(G) = \begin{cases} 1, & \text{if loops are forbidden and } G \cong \vec{C}_3; \\ 0, & \text{if } G \text{ is a bicomplete digraph.} \end{cases}$$

Theorem 5. *Let* $\mathrm{GDP}\,(B; G_1, , G_n)$ *satisfy following conditions:* (i) *Basis* B *has property* (D); (ii) *For* $i = 1, 2, \dots, n$, G_i *is a strongly connected digraph with at least two vertices;* (iii) *For* $i \in K = \{k | \exists \beta \in B \wedge \beta_k = -1\} \subset \{1, 2, \dots, n\}$, G_i *is a regular, non-complete digraph;* (iv) *For* $j \in L \subset \{1, 2, \dots, n\}$, G_j *is imprimitive with the index of imprimitivity* h_j, *otherwise it is primitive. The* $\mathrm{GDP}\,(B; G_1, \dots, G_n)$ *is a strongly connected digraph if and only if for every non-empty subset* $\{j_1, j_2, \dots, j_s\}$ *of* L, *which is consistent with the digraphs* $G_1, G_2, \dots, G_n$ *with respect to the basis* B *and every choice of integers* $\ell_{j_1}, \ell_{j_2}, \dots, \ell_{j_s}$, $1 \le \ell_{j_t} \le h_{j_t} - 1$, $t = 1, 2, \dots, s$, *there exists* $\beta \in B$ *such that*

$$\sum_{i \in \{j_1, \dots, j_s\}} \left(\frac{1}{2} ({\beta_i}^2 + \beta_i) \, \frac{\ell_i}{h_i} + \frac{\ell_i}{3} e(G_i) ({\beta_i}^2 - \beta_i) \right)$$

is not an integer.

Moreover, the number of strong components of GDP *is equal to the number of solutions* x_i, y_β *of the following system of equations*

$$\sum_{i \in L} \left(\frac{1}{2} ({\beta_i}^2 + \beta_i) \, \frac{x_i}{h_i} + \frac{x_i}{3} \, e(G_i) ({\beta_i}^2 - \beta_i) \right) = y_\beta, \quad \beta \in B,$$

satisfying condition: if for any $i \in L \cap K$, G_i *is not bicomplete or, in the case when loops are forbidden, neither bicomplete nor isomorphic to* $\vec{C}_3$, *then* $x_i = 0 (\operatorname{mod} h_i)$.

[1] l.c.m. denotes the lowest common multiple

Proof. According to Theorem 0.4 from [**2**] a digraph G, with an adjacency matrix A, is strongly connected if and only if its index r is a simple eigenvalue and if a positive eigenvector belongs to r both in A and A^{T}.

Let r_i be the index of G_i $(i = 1, 2, \ldots, n)$ and let x_i (y_i) be the positive eigenvector belonging to r_i in the adjacency matrix A_i (A_i^{T}) of G_i (theorem of Frobenius). Then, from Theorem 3, it immediately follows that $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ $(y_1 \otimes y_2 \otimes \cdots \otimes y_n)$ is the positive eigenvector belonging to the index Λ in the adjacency matrix A (A^T) of $G = \mathrm{GDP}\,(B; G_1, \ldots, G_n)$ where

$$\Lambda = \sum_{\beta \in B} r_1^{[\beta_1]} r_2^{[\beta_2]} \cdots r_n^{[\beta_n]}, \quad r_i^{[1]} = r_i, r_i^{[0]} = 1, r_i^{[-1]} = \nu_i p_i - r_i - \nu_i \cdot l(G_i),$$

p_i is the number of vertices, ν_i is the maximal number of parallel arcs between any two vertices or loops of a vertex of G_i and $l(G_i)$ have the same meaning as in Lemma 2.

By the same theorem, if none of $G_j, j \in K$, is complete, the index Λ of G can be obtained only from those eigenvalues of the digraphs G_j $(\overline{G_j})$, $j = 1, 2, \ldots, n$, which have a modulus equal to r_j $(\nu_j p_j - r_j - \nu_j \cdot l(G_j))$. All these eigenvalues of G_j can be written in the form $r_j \exp\left(\frac{\ell_j}{h_j} 2\pi\right)$, $0 \le \ell_j \le h_j - 1$ $(\exp(t) = e^{ti}, i^2 = -1)$ (theorem of Frobenius). Therefore, by Theorem 3 we have

$$\Lambda = \sum_{\beta \in B} \prod_{i=1}^{n} \left(\frac{1}{2}({\beta_i}^2 + \beta_i) r_i \exp\left(\frac{\ell_i}{h_i} 2\pi\right) + (1 - {\beta_i}^2) \right. \tag{2}$$
$$\left. + \frac{1}{2}({\beta_i}^2 - \beta_i) \left(\overline{\mathrm{sg}}\,(\ell_i) \nu_i p_i - \nu_i \cdot l(G_i) - r_i \exp\left(\frac{\ell_i}{h_i} 2\pi\right)\right)\right),$$

where $\overline{\mathrm{sg}}(0) = 1$ and $\overline{\mathrm{sg}}(x) = 0$ for $x > 0$. From (2) it follows that the index Λ is a simple eigenvalue if for each choice of integers ℓ_i, $0 \le \ell_i \le h_i - 1$, $i \in L$ with at least one $\ell_i > 0$, at least one summand in Λ is different from

$$r_1^{[\beta_1]} r_2^{[\beta_2]} \cdots r_n^{[\beta_n]}.$$

Let $N_n = \{1, 2, \ldots, n\}$. For any choice of integers $\ell_{j_1}, \ell_{j_2}, \ldots, \ell_{j_s}$, $1 \le \ell_{j_t} \le h_{j_t} - 1$, $\{j_1, j_2, \ldots, j_s\} \subset L$ and any $\beta \in B$ let $N_\beta = \{j_1, j_2, \ldots, j_s\} \cap \{k | \, \beta_k \ne 0\}$. Then from (2) we have:

$$\Lambda = \sum_{\beta \in B} \Bigl(\prod_{i \in N_n \setminus N_\beta} r_i^{[\beta_i]} \Bigr) \Bigl(\prod_{i \in N_\beta} \Bigl(\frac{1}{2}(1 + \beta_i) r_i \exp\left(\frac{\ell_i}{h_i} 2\pi\right)$$
$$+ \frac{1}{2}(1 - \beta_i) \Bigl(-\nu_i \cdot l(G_i) - r_i \exp\left(\frac{\ell_i}{h_i} 2\pi\right)\Bigr)\Bigr)\Bigr),$$

or

$$\Lambda = \sum_{\beta \in B} \Bigl(\prod_{i \in N_n \setminus N_\beta} r_i^{[\beta_i]} \Bigr) \Bigl(\prod_{i \in N_\beta} \Bigl({r_i}^2 + \frac{1}{2}(1 - \beta_i) \times \tag{3}$$

$$\times\Big(2r_i\nu_i\,\cos\frac{\ell_i}{h_i}2\pi+{\nu_i}^2\Big)l(G_i)\Big)^{\frac{1}{2}}\exp\,\Big(\frac{1}{2}(1+\beta_i)\frac{\ell_i}{h_i}2\pi+\frac{1}{2}(1-\beta_i)\Theta_i\Big)\Big)$$

$$\bigg(\Theta_i=\arg\Big(-\nu_i\cdot l(G_i)-r_i\exp\,\Big(\frac{\ell_i}{h_i}2\pi\Big)\Big)\bigg).$$

From (3) it follows that the index Λ of the GDP is a simple eigenvalue if and only if for at least one $\beta\in B$ one of the following conditions holds: (a) there exists $i\in N_\beta$ such that $\beta_i=-1$ and $\Big({r_i}^2+(2r_i\nu_i\,\cos\frac{\ell_i}{h_i}2\pi+{\nu_i}^2)\cdot l(G_i)\Big)^{\frac{1}{2}}$-$\neq\nu_ip_i-r_i-\nu_i\cdot l(G_i)$ or (b) the argument of the operator exp, of the corresponding summand in Λ, is different from $2k\pi$, $k\in\mathbf{Z}$.

By Lemma 2, we conclude that Λ is a simple eigenvalue whenever, in the case (a) the digraph G_i is not bicomplete or, in the case when loops are forbidden, neither bicomplete nor isomorphic to $\vec{C}_3$.

Using condition (b) and supposing that, in accordance with (a), for each $i\in N_\beta$ for which $\beta_i=-1$, G_i is bicomplete or, in the case when loops are forbidden, bicomplete or isomorphic to $\vec{C}_3$, we have condition

$$\sum_{i\in N_\beta}\Big(\frac{1}{2}\,(1+\beta_i)\frac{\ell_i}{h_i}2\pi+\frac{1}{2}\,(1-\beta_i)\frac{\ell_i}{3}\,4\pi e(G_i)\Big)\neq 2k\pi,$$

from which the first part of the Theorem follows.

According to Theorem 0.5 from **[2]** the number of strong components of the GDP is equal to the multiplicity of its index Λ. For any choice of integers $\ell_{j_1},\ell_{j_2},\dots,\ell_{j_s}$, $\{j_1,j_2,\dots,j_s\}\subset L$, $1\le\ell_{j_t}\le h_{j_t}-1$, $t=1,2,\dots,s$, let G_{j_t} is bicomplete or, in the case when loops are forbidden, bicomplete or isomorphic to $\vec{C}_3$ whenever $j_t\in K\cap L$ holds. Then (3), according to Lemma 2, can be written in the form:

$$\Lambda=\sum_{\beta\in B}r_1^{[\beta_1]}r_2^{[\beta_2]}\cdots r_n^{[\beta_n]}\times \tag{4}$$

$$\times\exp\,\bigg(\sum_{i\in\{j_1,\dots,j_s\}}\Big(\frac{1}{2}\,({\beta_i}^2+\beta_i)\frac{\ell_i}{h_i}2\pi+\frac{1}{2}\,({\beta_i}^2-\beta_i)\frac{\ell_i}{3}4\pi e(G_i)\Big)\bigg).$$

Now (4) gives Λ if the argument of the operator exp, in all summands, is equal to $2y_\beta\pi$, $y_\beta\in\mathbf{Z}$.

This completes the proof of the theorem.

The following theorem is a specialization of the preceding one to undirected graphs. Preliminary we introduce the following function.

$$f(x_1,x_2,\dots,x_n)=\sum_{\beta\in B}x_1^{[\beta_1]}x_2^{[\beta_2]}\cdots x_n^{[\beta_n]},\ x_i^{[1]}=x_i,\ x_i^{[0]}=1,\ x_i^{[-1]}={x_i}^2. \tag{5}$$

If (see **[2]** for definition) the function (5) is even (odd) with respect to a non-empty subset $\{x_{i_1},x_{i_2},\dots,x_{i_s}\}$ of variables $\{x_1,x_2,\dots,x_n\}$ ($\{i_1,i_2,\dots,i_s\}\subset$

$\{1, 2, \ldots, n\}$) we will say also that this function is even (odd) with respect to the subset $\{i_1, i_2, \ldots, i_s\}$ of $\{1, 2, \ldots, n\}$.

Theorem 6. *Let* $GDP(B; G_1, \ldots, G_n)$ *satisfy following conditions*: (i) *Basis* B *has property* (D), (ii) *For* $i = 1, 2, \ldots, n$, G_i *is an undirected connected graph with at least two vertices*; (iii) *For* $i \in K = \{k | \exists\beta \in B \wedge \beta_k = -1\} \subset \{1, 2, \ldots, n\}$, G_i *is a regular non-complete graph*; (iv) *For* $j \in L \subset \{1, 2, \ldots, n\}$, G_j *is a bipartite graph, otherwise it is primitive. The* $\mathrm{GDP}(B; G_1, \ldots, G_n)$ *is a connected graph if and only if the function* (5) *is never even with respect to a non-empty subset of* L, *which is consistent with the graphs* $G_1, G_2, \ldots, G_n$ *with respect to the basis* B.

Moreover, the number of components of $\mathrm{GDP}(B; G_1, \ldots, G_n)$ *is greater by one than the number of non-empty subsets of* L, *which are consistent with the graphs* $G_1, G_2, \ldots, G_n$ *with respect to the basis* B, *with respect to which the function* (5) *is even.*

Example 1. The following system of equations

$$\frac{x_1}{h_1} + \frac{x_2}{h_2} = x \,\&\, \frac{x_1}{h_1} + \frac{2x_2}{h_2} e(G_2) = y \,\&\, \frac{2x_1}{h_1} e(G_1) + \frac{x_2}{h_2} = z.$$

or, in the case of undirected graphs, the function

$$f(x_1, x_2) = x_1 x_2 + {x_1}^2 x_2 + x_1 {x_2}^2$$

is corresponded to the generalized direct product with basis $B = \{(1,1), (1,-1), (-1,1)\}$ of regular, connected, non-complete digraphs G_1 and G_2 each containing at least two vertices. There are no solutions of this system of equations different from zero, which satisfy conditions of Theorem 5, and therefore, the considered product is connected. In the case when any factor is complete, it is easy to see that this product is also connected.

Product in this example is corresponded to the Boolean function disjunction of graphs (see [**2**], p. 207).

Example 2. Generalized direct product with basis $B = \{(1,1), (-1,-1)\}$ of regular, connected, non-complete digraphs G_1, G_2, each containing at least two vertices, has two components if digraphs G_1 and G_2 are bicomplete, three components (each isomorphic to C_3) if these two digraphs are isomorphic to $\vec{C}_3$ and loops are not allowed. In other cases this product is connected. If any of the factors is complete this product is reduced to the ordinary product of digraphs and is a connected digraph.

Example 3. Generalized direct product with basis $B = \{(1,-1), (-1,1)\}$ of regular, connected, non-complete digraphs G_1, G_2, each containing at least two vertices is always a connected digraph except in the case that both of the factors are isomorphic to $\vec{C}_3$ and loops are not allowed. In this case this GDP has three components (each isomorphic to C_3). If one of the factors is complete this product is connected if and only if the complement of the other factor is connected.

Results related to the connectedness of the NEPS and its special cases given in [**3**] and [**5**] follow now immediately from Theorem 5.

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