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ON SOME INTEGRALS INVOLVING THE RIEMANN ZETA-FUNCTION IN THE CRITICAL STRIP

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For $1/2 < \sigma < 3/4$ let $E_\sigma(T)$ denote the error term in the asymptotic formula for $\int_0^T |\zeta(\sigma + it)|^2 \mathrm{d}t$. If $U_\sigma(T)$ and $V_\sigma(T)$ denote the error terms in the asymptotic formulas for $\int_0^T E_\sigma(t)|\zeta(\sigma+it)|^2 \,\mathrm{d}t$ and $\int_0^T E_\sigma(t)^2|\zeta(\sigma+it)|^2 \,\mathrm{d}t$ respectively, then we obtain some asymptotic results on $U_\sigma(T)$ and $V_\sigma(T)$.

1. INTRODUCTION

Recently much work has been done on the function

$$E_\sigma(t) = \int_0^T |\zeta(\sigma + it)|^2 \,\mathrm{d}t - \zeta(2\sigma)T - \frac{(2\pi)^{2\sigma-1}\zeta(2-2\sigma)}{2-2\sigma}\, T^{2-2\sigma}, \tag{1}$$

where $1/2 < \sigma < 1$ is fixed. This important function, which represents the error term in the mean square formula for the RIEMANN zeta–function $\zeta(s)$, in the so--called "critical strip" $1/2 < \sigma\ (= \operatorname{Re} s) < 1$, was introduced by K. MATSUMOTO [**6**]. He obtained several results, including an upper bound for the function

$$F_\sigma(T) = \int_0^T E_\sigma(t)^2 \,\mathrm{d}t - c(\sigma)T^{5/2-2\sigma}, \tag{2}$$

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where

$$(3)\quad c(\sigma) = \frac{2(2\pi)^{2\sigma-3/2}}{5-4\sigma}\cdot\frac{\zeta^2(3/2)\,\zeta(5/2-2\sigma)\,\zeta(1/2+2\sigma)}{\zeta(3)}\qquad (1/2<\sigma<3/4).$$

His work is continued in Matsumoto-Meurman [**9**], where it was proved that

$$(4)\qquad F_\sigma(T) = O(T)\qquad (1/2<\sigma<3/4),$$

$$\int_0^T E_{3/4}(t)^2\,\mathrm{d}t = \frac{\zeta^2(3/2)\,\zeta(2)}{\zeta(3)}\,T\log T + O\left(T(\log T)^{1/2}\right),$$

and

$$\int_0^T E_\sigma(t)^2\,\mathrm{d}t = O(T)\qquad (3/4<\sigma<1).$$

Some further results are obtained by Ivić–Matsumoto [**4**], and an extensive survey on the known results on $E_\sigma(T)$ is given by K. Matsumoto [**7**].

Recently the second author [**5**] proved that

$$(5)\quad \int_0^T E_\sigma(t)^2|\zeta(\sigma+it)|^2\,\mathrm{d}t = \begin{cases} c_1c_3T^{5/2-2\sigma} + c_5T^{7/2-4\sigma} + O(T) & (1/2<\sigma<5/8),\\ c_1c_3T^{5/2-2\sigma} + O(T) & (5/8\le\sigma<3/4),\\ c_4\zeta(3/2)T\log T + O\left(T(\log T)^{1/2}\right) & (\sigma=3/4),\\ O(T) & (3/4<\sigma<1),\end{cases}$$

where $c_1 = c_1(\sigma) = \zeta(2\sigma)$, $c_3(\sigma) = c(\sigma)$ is given by (3),

$$(6)\qquad c_4 = \frac{\zeta^2(3/2)\,\zeta(2)}{\zeta(3)},\quad c_5 = c_5(\sigma) = \frac{5-4\sigma}{7-8\sigma}\,(2\pi)^{2\sigma-1}\,\zeta(2-2\sigma)\,c(\sigma).$$

The above results are the analogues of the results obtained by the first author ([**2**] and [**3**], Ch.3) for the integrals

$$\int_0^T E(t)^k\left|\zeta\left(\frac12+it\right)\right|^2\,\mathrm{d}t$$

in the case $k=2$. Here, as usual (γ is Euler's constant),

$$E(T) = \int_0^T\left|\zeta\left(\frac12+it\right)\right|^2\,\mathrm{d}t - T\left(\log\frac{T}{2\pi}+2\gamma-1\right)$$

is the error term in the asymptotic formula for the mean square of $\zeta(s)$ on the "critical line" $\sigma = \operatorname{Re} s = 1/2$.

The aim of this note is to sharpen (5) in the range $1/2 < \sigma < 3/4$ and to investigate a related integral. To this end we define

$$U_\sigma(T) = \int_0^T E_\sigma(t)\,|\zeta(\sigma+it)|^2\,\mathrm{d}t - c_1(\sigma)B(\sigma)T - c_2(\sigma)B(\sigma)T^{2-2\sigma} \tag{7}$$

and

$$V_\sigma(T) = \int_0^T E_\sigma(t)^2|\zeta(\sigma+it)|^2\mathrm{d}t - c_1(\sigma)c_3(\sigma)T^{5/2-2\sigma} - c_5(\sigma)T^{7/2-4\sigma}, \tag{8}$$

where $c_1(\sigma) = \zeta(2\sigma)$, $B(\sigma) = -2\pi\zeta(2\sigma-1)$, $c_3(\sigma) = c(\sigma)$ is given by (3), $c_5(\sigma)$ by (6), and

$$c_2(\sigma) = \frac{(2\pi)^{2\sigma-1}\zeta(2-2\sigma)}{2-2\sigma}\,. \tag{9}$$

Then we shall prove

Theorem 1. *For fixed σ such that $1/2 < \sigma < 3/4$ and $U_\sigma(T)$ defined by (7) we have*

$$U_\sigma(T) = O\left(T^{5/4-\sigma}\right), \qquad U_\sigma(T) = \Omega_\pm\left(T^{5/4-\sigma}\right), \tag{10}$$

$$\int_0^T U_\sigma(t)\,\mathrm{d}t = \frac{1}{2}\,c(\sigma)T^{5/2-2\sigma} + O\left(T^{7/4-\sigma}\right), \tag{11}$$

and

$$\int_0^T U_\sigma(t)^2\,\mathrm{d}t = c_1(\sigma)^2c_6(\sigma)T^{7/2-2\sigma} + O\left(T^{9/2-4\sigma} + T^{(11-8\sigma)/3}\right), \tag{12}$$

where

$$c_6 = \frac{4^{\sigma-1}\pi^{2\sigma-3}\sqrt{2\pi}}{7-4\sigma}\sum_{n=1}^{+\infty}\sigma_{1-2\sigma}(n)^2n^{2\sigma-7/2}, \qquad \sigma_z(n) = \sum_{d|n} d^z \quad (z\in\mathbf{C}). \tag{13}$$

Theorem 2. *For fixed σ such that $1/2 < \sigma < 3/4$, $V_\sigma(T)$ defined by (8) and $F_\sigma(T)$ defined by (2) we have*

$$V_\sigma(T) = \zeta(2\sigma)F_\sigma(T) + O\left(T^{2-2\sigma}\right). \tag{14}$$

We remark that (14) coupled with (4) implies the first two formulas in (5). We recall that $f(x) = \Omega_\pm(g(x)) \quad (g(x) > 0 \text{ for } x \ge x_0)$ means that both

$$\liminf_{x\to\infty} \frac{f(x)}{g(x)} < 0 \quad \text{and} \qquad \limsup_{x\to\infty} \frac{f(x)}{g(x)} > 0$$

hold.

2. PROOF OF THEOREM 1

Let $0 \le H \le T$, $f(t) \in C[T, T+H]$, and $F' = f$. Then from (1) it follows that

$$\int_T^{T+H} f(E_\sigma(t))\,|\zeta(\sigma+it)|^2\,\mathrm{d}t \tag{15}$$

$$= \int_T^{T+H} f(E_\sigma(t))\,\mathrm{d}E_\sigma(t) + \int_T^{T+H} f(E_\sigma(t))\Big(\zeta(2\sigma) + (2\pi)^{2\sigma-1}\zeta(2-2\sigma)t^{1-2\sigma}\Big)\mathrm{d}t$$

$$= F(E_\sigma(T+H)) - F(E_\sigma(T)) + \int_T^{T+H} f(E_\sigma(t))\Big(\zeta(2\sigma) + (2\pi)^{2\sigma-1}\zeta(2-2\sigma)t^{1-2\sigma}\Big)\mathrm{d}t.$$

We shall apply (15) with $H = T,\ f(t) = t,\ F(t) = t^2/2$, and use the notation

$$\int_0^T E_\sigma(t)\,\mathrm{d}t = B(\sigma)T + G_\sigma(T). \tag{16}$$

The function $G_\sigma(T)$ was introduced by the first author in [**3**], Ch. 3 (the lower bound of integration in [**3**] was 2, but this is of no consequence). He showed that

$$G_\sigma(T) = O\left(T^{5/4-\sigma}\right), \quad G_\sigma(T) = \Omega_\pm\left(T^{5/4-\sigma}\right), \tag{17}$$

thereby determining the true order of magnitude of $G_\sigma(T)$. The constant $B(\sigma)$ was explicitly given in [**3**] by a rather complicated expression, which was simplified in Matsumoto–Meurman [**9**] to $B(\sigma) = -2\pi\zeta(2\sigma-1)$. Thus from (15) we obtain, with the aid of (16),

$$\int_T^{2T} E_\sigma(t)\,|\zeta(\sigma+it)|^2\,\mathrm{d}t = \frac{1}{2}\,E_\sigma(2T)^2 - \frac{1}{2}\,E_\sigma(T)^2$$

$$+\int_T^{2T}\Big(\zeta(2\sigma)+(2\pi)^{2\sigma-1}\zeta(2-2\sigma)t^{1-2\sigma}\Big)\Big(B(\sigma)\,\mathrm{d}t+\mathrm{d}G_\sigma(t)\Big)$$

$$=\Big(\frac12\,E_\sigma(t)^2+\zeta(2\sigma)B(\sigma)t+c_2(\sigma)B(\sigma)\,t^{2-2\sigma}+\zeta(2\sigma)G_\sigma(t)$$

$$+\ (2\pi)^{2\sigma-1}\zeta(2-2\sigma)\,t^{1-2\sigma}G_\sigma(t)\Big)\Big|_T^{2T}$$

$$+(2\sigma-1)(2\pi)^{2\sigma-1}\zeta(2-2\sigma)\int_T^{2T}G_\sigma(t)t^{-2\sigma}\,\mathrm{d}t.$$

Replacing T by $T/2^j$ and summing over $j=1,2,\ldots$, we obtain, in view of (7),

$$U_\sigma(T)=\frac12\,E_\sigma(T)^2+\zeta(2\sigma)G_\sigma(T)+(2\pi)^{2\sigma-1}\zeta(2-2\sigma)T^{1-2\sigma}G_\sigma(T)\tag{18}$$

$$+(2\sigma-1)(2\pi)^{2\sigma-1}\zeta(2-2\sigma)\int_1^T G_\sigma(t)\,t^{-2\sigma}\,\mathrm{d}t+O(1).$$

From (18) we shall deduce now all the results of Theorem 1. In [**3**] it was proved that, for $1/2<\sigma<1$, $E_\sigma=O(T^{1-\sigma})$. This was improved in [**4**] to $E_\sigma(t)=O\left(T^{2(1-\sigma)/3}\log^{2/9}T\right)$, but by combining several estimates which come from the use of the theory of exponent pairs one can dispose of the log-factor and obtain in fact

$$E_\sigma(T)=O\left(T^{2(1-\sigma)/3}\right)\qquad(1/2<\sigma<1).\tag{19}$$

Hence from (17)–(19) we easily obtain (10), since $G_\sigma(T)$ is the dominant term in (18).

To obtain (11) we use Theorem 3.2 of [**3**], which states that

$$G_\sigma(T)=2^{\sigma-2}\left(\frac{\pi}{T}\right)^{\sigma-\frac12}\sum_{n\le N}(-1)^n\sigma_{1-2\sigma}(n)n^{\sigma-1}\cdot\tag{20}$$

$$\cdot\left(\operatorname{arsinh}\sqrt{\frac{\pi n}{2T}}\right)^{-2}\left(\frac{T}{2\pi n}+\frac14\right)^{-1/4}\sin(f(T,n))$$

$$-\ 2\left(\frac{2\pi}{T}\right)^{\sigma-\frac12}\sum_{n\le N'}\sigma_{1-2\sigma}(n)^{\sigma-1}n^{\sigma-1}\left(\log\frac{T}{2\pi n}\right)^{-2}\sin(g(T,n))+O\left(T^{3/4-\sigma}\right),$$

where, for arbitrary fixed constants $0<A<A'$, $AT<N<A'T$,

$$N'\ =\ \frac{T}{2\pi}+\frac{N}{2}-\left(\frac{N^2}{4}+\frac{NT}{2\pi}\right)^{1/2},$$

$$
\begin{aligned}
f(T,n) &= 2T\operatorname{arsinh}\sqrt{\frac{\pi n}{2T}} + (2\pi nT + \pi^2 n^2)^{1/2} - \frac{\pi}{4}\,,\\
g(T,n) &= T\log\frac{T}{2\pi n} - T + \frac{\pi}{4}\,,
\end{aligned}
$$

and

$$
\operatorname{arsinh} x = \log\left(x + \sqrt{x^2+1}\right).
$$

The contribution of the error term in (20) is estimated trivially, and for the sums over n we use the first derivative test (Lemma 2.1 of [**1**]) to obtain

$$
\int_T^{2T} G_\sigma(t)\,\mathrm{d}t = O\left(T^{7/4-\sigma}\right), \qquad \int_T^{2T} t^{1-2\sigma} G_\sigma(t)\,\mathrm{d}t = O\left(T^{11/4-3\sigma}\right),
$$

and then by integration by parts and the first of the above estimates

$$
\int_1^T G_\sigma(t)\,t^{-2\sigma}\,\mathrm{d}t = O\left(T^{7/4-3\sigma} + \log T\right).
$$

It follows from (2), (4), (18) and the above estimates that

$$
\int_T^{2T} U_\sigma(t)\,\mathrm{d}t = \frac{1}{2}\,c(\sigma)T^{5/2-2\sigma} + O\left(T^{7/4-\sigma}\right),
$$

which in turn implies (11).

It is not unreasonable to conjecture that

$$
\int_0^T U_\sigma(t)\,\mathrm{d}t - \frac{1}{2}\,c(\sigma)T^{5/2-2\sigma} = \Omega_\pm\left(T^{7/4-\sigma}\right). \tag{21}
$$

The omega–result in (21) would be an analogue of $V(T) = \Omega_\pm(T^{5/4}\log T)$ of Theorem 3.7 of [**3**]. However, (21) appears to be more difficult and at present we cannot prove it. The reason for this is essentially the appearance of "$\log T$" in the definition of $E(T)$, while there is no logarithm in the definition (1) of $E_\sigma(T)$, although $E(T) = \lim\limits_{\sigma\to\frac{1}{2}+0} E_\sigma(T)$. Analysing the proof of Theorem 3.7 of [**3**] it will be clear that it is precisely the appearance of $\log T$ in front of the series in (3.99) of [**3**] which makes it possible to deduce that $V(T) = \Omega_\pm(T^{5/4}\log T)$, and such an argument is not applicable in the case of (21).

To prove (12) note first that from (18) we have

$$
U_\sigma(t) = \frac{1}{2}\,E_\sigma(t)^2 + \zeta(2\sigma)G_\sigma(t)\Big(1 + O\big(t^{1-2\sigma}\big)\Big) + O\left(t^{7/4-3\sigma} + \log t\right). \tag{22}
$$

This gives

$$\int_T^{2T} U_\sigma(t)^2\,\mathrm{d}t = \Big(\zeta^2(2\sigma) + O\big(T^{1-2\sigma}\big)\Big)\int_T^{2T} G_\sigma(t)^2\,\mathrm{d}t + \frac{1}{4}\int_T^{2T} E_\sigma(t)^4\,\mathrm{d}t \tag{23}$$

$$+\ \zeta(2\sigma)\int_T^{2T} G_\sigma(t)E_\sigma(t)^2\Big(1+O\big(t^{1-2\sigma}\big)\Big)\,\mathrm{d}t$$

$$+\ O\left(\int_T^{2T}\big(|G_\sigma(t)|+|E_\sigma(t)|^2\big)\big(t^{7/4-3\sigma}+\log t\big)\,\mathrm{d}t\right)$$

$$+\ O\Big(T^{9/2-6\sigma}+T\log^2 T\Big).$$

We shall use now Theorem 3.5 of [**3**] which says that

$$\int_2^{T} G_\sigma(t)^2\,\mathrm{d}t = c_6(\sigma)T^{7/2-2\sigma} + O\big(T^{3-2\sigma}\big) \tag{24}$$

with $c_6(\sigma)$ given by (13). It follows from (2), (4) and (19) that

$$\int_T^{2T} E_\sigma(t)^4\,\mathrm{d}t \ll \max_{T\le t\le 2T}|E_\sigma(t)|^2\int_T^{2T} E_\sigma(t)^2\,\mathrm{d}t \ll T^{23/6-10\sigma/3}.$$

Thus by the Cauchy–Schwarz inequality, we have

$$\int_T^{2T} G_\sigma(t)E_\sigma(t)^2\,\mathrm{d}t \ll T^{(11-8\sigma)/3}$$

and

$$\int_T^{2T}\big(|G_\sigma(t)|+|E_\sigma(t)|^2\big)\big(t^{7/4-3\sigma}+\log t\big)\,\mathrm{d}t \ll T^{4-4\sigma}+T^{9/4-\sigma}\log T.$$

Consequently from (23) and (24) we obtain

$$\int_T^{2T} U_\sigma(t)^2\,\mathrm{d}t\ =\ \zeta^2(2\sigma)c_6(\sigma)\,t^{7/2-2\sigma}\Big|_T^{2T}$$

$$+\ O\Big(T^{(11-8\sigma)/3}+T^{9/2-4\sigma}+T^{9/4-\sigma}\log T+T^{4-4\sigma}\Big).$$

The largest of the exponents in the error term is $9/2-4\sigma$, $(11-8\sigma)/3$ for $1/2<\sigma\le 5/8$, $5/8\le\sigma<3/4$, respectively, and therefore (12) follows.

3. PROOF OF THEOREM 2

We start from (15) with $H = T$, $f(t) = t^2$ and $F(t) = t^3/3$. Using (2) and (19) we obtain

$$\begin{aligned}
&\int_T^{2T} E_\sigma(t)^2|\zeta(\sigma+it)|^2\,\mathrm{d}t \\
&= \frac{1}{3}\,E_\sigma(2T)^3 - \frac{1}{3}\,E_\sigma(T)^3 + \int_T^{2T}\Big(\zeta(2\sigma) + (2\pi)^{2\sigma-1}\zeta(2-2\sigma)t^{1-2\sigma}\Big)E_\sigma(t)^2\,\mathrm{d}t \\
&= O\big(T^{2-2\sigma}\big) + \int_T^{2T}\Big(\zeta(2\sigma) + (2\pi)^{2\sigma-1}\zeta(2-2\sigma)t^{1-2\sigma}\Big)\cdot \\
&\qquad\qquad\cdot\Big((5/2-2\sigma)c(\sigma)t^{3/2-2\sigma}\,\mathrm{d}t + \mathrm{d}F_\sigma(t)\Big) \\
&= O\big(t^{2-2\sigma}\big) + \Big(\zeta(2\sigma)c(\sigma)t^{5/2-2\sigma} + \frac{(5-4\sigma)(2\pi)^{2\sigma-1}\zeta(2-2\sigma)c(\sigma)}{7-8\sigma}\,t^{7/2-4\sigma} \\
&\qquad + \zeta(2\sigma)F_\sigma(t) + (2\pi)^{2\sigma-1}\zeta(2-2\sigma)\,t^{1-2\sigma}F_\sigma(t)\Big)\Big|_T^{2T} \\
&\qquad + (2\sigma-1)(2\pi)^{2\sigma-1}\zeta(2-2\sigma)\int_T^{2T} F_\sigma(t)\,t^{-2\sigma}\,\mathrm{d}t.
\end{aligned}$$

This holds for $1/2 < \sigma < 1$, but using (4) in the range $1/2 < \sigma < 3/4$ we obtain

$$V_\sigma(2T) - V_\sigma(T) = \zeta(2\sigma)(F_\sigma(2T) - F_\sigma(T)) + O\big(T^{2-2\sigma}\big), \tag{25}$$

and from (25) we easily obtain (14). Lack of bounds for the integrals of $F_\sigma(T)$ and $F_\sigma(T)^2$ precludes the derivation of the analogues of (11) and (12) for $V_\sigma(T)$.

We conclude by making some remarks and conjectures. Note that Matsumoto–Meurman [**9**] proved, for $1/2 < \sigma < 3/4$, that

$$F_\sigma(T) = \Omega\Big(T^{9/4-3\sigma}(\log T)^{3\sigma-3/4}\Big), \tag{26}$$

where as usual $f(x) = \Omega(g(x))$ $\quad(g(x) > 0$ for $x \ge x_0)$ means that

$$\limsup_{x\to\infty}\frac{|f(x)|}{g(x)} > 0.$$

They also conjectured, for $1/2 < \sigma < 3/4$, that as $T \to \infty$

$$F_\sigma(T) = \big(B(\sigma)^2 + o(1)\big)\,T. \tag{27}$$

If (27) is true, then from (14) it follows that, as $T \to \infty$,

$$V_\sigma(T) = \big(\zeta(2\sigma)B(\sigma)^2 + o(1)\big)\,T \qquad (1/2 < \sigma < 3/4). \tag{28}$$

Perhaps, if (27) is not true, then in view of (26) one has

$$V_\sigma(T) = \Omega\Big(T^{9/4-3\sigma}(\log T)^{3\sigma-3/4}\Big) \qquad (1/2 < \sigma < 3/4). \tag{29}$$

Note that, in view of (16) and (17), $B(\sigma)$ is the mean value of $E_\sigma(T)$. Hence it seems to make sense to evaluate also

$$\begin{aligned} K_\sigma(T) &:= \int_0^T (E_\sigma(t) - B(\sigma))^2\,|\zeta(\sigma+it)|^2\,\mathrm{d}t \\ &= \int_0^T E_\sigma(t)^2|\zeta(\sigma+it)|^2\,\mathrm{d}t - 2B(\sigma)\int_0^T E_\sigma(t)\,|\zeta(\sigma+it)|^2\,\mathrm{d}t \\ &+ B(\sigma)^2\int_0^T |\zeta(\sigma+it)|^2\,\mathrm{d}t. \end{aligned}$$

Using (1), (7) and (8) we can write

$$\begin{aligned} K_\sigma(T) &= c_1(\sigma)c_3(\sigma)T^{5/2-2\sigma} + c_5(\sigma)T^{7/2-4\sigma} - c_1(\sigma)B(\sigma)^2T \\ &- c_2(\sigma)B(\sigma)^2T^{2-2\sigma} + V_\sigma(T) - 2B(\sigma)U_\sigma(T) + B(\sigma)^2E_\sigma(T). \end{aligned}$$

Using (14), (19) and (22) we obtain then

$$\begin{aligned} K_\sigma(T) &= c_1(\sigma)c_3(\sigma)T^{5/2-2\sigma} + c_5(\sigma)T^{7/2-4\sigma} - c_1(\sigma)B(\sigma)^2T \\ &-c_2(\sigma)B(\sigma)^2T^{2-2\sigma} + c_1(\sigma)F_\sigma(T) - 2c_1(\sigma)B(\sigma)G_\sigma(T)\Big(1+O\left(T^{1-2\sigma}\right)\Big) + O(T^{2-2\sigma}). \end{aligned} \tag{30}$$

If (27) holds then (30) yields, as $T\to\infty$,

$$K_\sigma(T) = c_1(\sigma)c_3(\sigma)T^{5/2-2\sigma} + c_5(\sigma)T^{7/2-4\sigma} + o(T). \tag{31}$$

It appears to us that perhaps the error term in (31) is $O\left(T^{2-2\sigma}\right)$. The viewpoint is also shared by K. Matsumoto who, in correspondence, has kindly made several remarks concerning the true order of the functions $F_\sigma(T)$ and $K_\sigma(T)$. In particular, he pointed out that, in view of his recent work [**8**], it is quite plausible that

$$F_\sigma(T) = B(\sigma)^2T + O\left(T^{2-2\sigma}\log^C T\right) \qquad (1/2 < \sigma < 3/4,\ \ C \ge 0) \tag{32}$$

holds, perhaps even with $C = 0$. Note that (31) is stronger than (27). If it is true, then (30) gives

$$K_\sigma(T) = c_1(\sigma)c_3(\sigma)\,T^{5/2-2\sigma} + c_5(\sigma)\,T^{7/2-4\sigma} + O\left(T^{2-2\sigma}\log^C T\right).$$

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