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A SEARCH FOR 4-DESIGNS ARISING BY ACTION OF $PGL(2,q)$[1]

Dragan M. Acketa, Vojislav Mudrinski, Đura Paunić

A complete search for 4-$(q+1,5,\lambda)$ designs arising by action of groups $PGL(2,q)$ is made, for all prime powers $q \leq 32$. The search is based on the use of (λ_{ij}) matrices, which are constructed by using 3-homogenicity of this action. The elements of these matrices are numbers of inclusions of 4-subsets of the ground-set within 5-subsets, partitioned w.r.t. the orbits. It turns out that there exist only eight designs of the considered type, two for $q = 17$ and six for $q = 32$.

1. INTRODUCTION AND CONSTRUCTION

An $n-$set is a set of cardinality n. A t-(v,k,λ) *design* [**3**] is an incidence structure on v points, which consists of some k-sized sets of points (called *blocks*) without repetitions and satisfies the property that each t points are contained in exactly λ blocks. As usual, $GF(q)$ denotes the Galois field associated to a prime power q. $PGL(2,q)$ denotes the group of projective linear transformations over $(GF(q))^2$. Each element of $PGL(2,q)$ can be represented by a regular 2×2 matrix of the form $M = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$, which acts as follows:

$$(\overline{x}_1, \overline{x}_2) = (x_1, x_2) \cdot \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \qquad \text{(equivalently, } \overline{x} = x^M \text{)},$$

where (a_1,a_2), $x = (x_1,x_2)$ and $\overline{x} = (\overline{x}_1, \overline{x}_2)$ are general elements of $(GF(q))^2$, while (b_1,b_2) is required to belong to the canonical set

$$C(q) = \{(0,1),\ (1,1), \ldots,\ (q-1,1),\ (1,0)\}.$$

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It is known that $PGL(2,q)$ acts 3-transitively on the ground-set

$$V(q) = \{0,1,\ldots,q-1\} \cup \{\infty\}.$$

Consider the mapping $\beta : V(q) \longrightarrow C(q)$ defined by $x^\beta = (x,1)$ for $x \in \{0,1,\ldots,q-1\}$ and $\infty^\beta = (1,0)$. The image $\overline{x} \in V(q)$ of an element $x \in V(q)$ under an element M of $PGL(2,q)$ is determined as $\overline{x} = x^{\beta M \gamma \beta^{-1}}$, where γ maps each non-zero element y of $(GF(q))^2$ onto the unique element of $C(q)$, which belongs to the 1-dimensional subspace determined by y.

The group $PGL(2,q)$ can be also represented by an array of size $(q^3-q)\times q$, each row of which is a permutation of $V(q)$. This representation turned out to be inefficient in implementation of the algorithm, it was tried and abandoned. The 2×2 matrix representation of elements of $PGL(2,q)$, mentioned in the beginning of the paper, was used for the computations.

Let T and B denote the families of all those subsets of $V(q)$, which are of cardinalities 4 and 5 respectively and let $T_1,\ldots,T_m$ and $B_1,\ldots,B_n$ denote the orbits of T and B by action of $PGL(2,q)$.

It is easy to show for each $i \in \{1,\ldots,m\}$ and for each $j \in \{1,\ldots,n\}$ (details can be found in [**1**) that each 4-subset of T_i is contained into the same number (denoted by λ_{ij}) of 5-subsets of B_j. Conversely, each 5-subset of B_j is contained into the same number (denoted by h_{ij}) of 4-subsets of T_i. It holds that $\lambda_{i1}+\cdots+\lambda_{in} = q-3 = \lambda$ of the trivial design, for each $i \in \{1,\ldots,m\}$ and $h_{1j}+\cdots+h_{mj} = 5$, for each $j \in \{1,\ldots,n\}$.

The 3-homogenicity of the action of $PGL(2,q)$ enables a reduction of the search for orbits of 4-subsets and 5-subsets of $V(q)$ to those of these subsets, which contain a fixed 3-subset S. This leads to an efficient method (described in more details in [**1**]) for computing the (λ_{ij}) matrix. When looking for the (h_{ij}) matrix, the considerations should also include those 4-subsets of $V(q)$, which have 2-intersections with S. The i-th row of the (λ_{ij}) matrix (the j-th column of the (h_{ij}) matrix) can be calculated by considering the partition w.r.t. orbits of 5-supersets (4-subsets) of a representative of T_i $(B_j.)$

Counting the incidencies between 4-subsets in T_i and 5-subsets in B_j in two different manners, it immediately follows that $|T_i|\cdot\lambda_{ij} = |B_j|\cdot h_{ij}$. This relationship enables the computation of one of the two matrices from the other, provided that the orbit cardinalities are known.

Example. The (h_{ij}) matrix for $q = 32$, with the same order of orbits as with the (λ_{ij}) matrix in Table 1., has the following outlook:

4	1	2	1	1	1	1	0	0	0	0
1	1	0	2	1	1	0	4	1	0	0
0	0	1	1	1	1	4	0	2	1	0
0	2	1	1	1	0	0	0	1	4	1
0	1	1	0	1	2	0	1	1	0	4

Each one of the five orbits of 4-subsets is of cardinality 267. The 1st, the 7th, the 8th, the 10th and the 11th orbit of 5-subsets is of cardinality 15 (these five orbits will have a special role in Section 2), while the remaining six orbits of 5-subsets are of cardinality 60.

Having the (λ_{ij}) matrix, the *construction* itself runs as follows:

A search for those non-trivial 4-designs is made, the blocks of which are all the 5-subsets of orbits $B_{j_1}, \ldots, B_{j_s}$, for some proper subset $\{j_1, \ldots, j_s\}$ of $\{1, \ldots, n\}$. The condition for existence of such a 4-design is that the sums $\lambda_{ij_1} + \cdots + \lambda_{ij_s}$ are equal for all $i \in \{1, \ldots, m\}$; if this is the case, then the common value of these sums is λ of the design.

Thus a non-trivial 4-design corresponds to a proper subset P of the column set of the (λ_{ij}) matrix, with the property that all the row sums of the submatrix determined by P are equal. A search for such a submatrix can be performed, e.g., by applying the Gray code to the column set.

2. RESULTS

The results of the performed computer search for 4-designs, which arise by action of $PGL(2,q)$, can be summarized as follows:

Theorem 1. *Action of the group $PGL(2,q)$ for prime powers $q \leq 32$ gives exactly eight 4-$(q+1,5,\lambda)$ designs, for the pairs (q,λ) equal to* $(17,4)$, $(17,10)$, $(32,4)$, $(32,5)$, $(32,9)$, $(32,20)$, $(32,24)$ *and* $(32,25)$.

Proof. The (λ_{ij}) matrices for $q \leq 32$, obtained by the above method, are shown in Table 1. The value of q is given in the upper left direction with respect to a matrix. A row of a (λ_{ij}) matrix denoted by (a) corresponds to the orbit of 4-sets having the representative $\{0,1,a,\infty\}$. Similarly, a column denoted by (a,b) corresponds to the orbit of 5-sets having the representative $\{0,1,a,b,\infty\}$:

Table 1. (λ_{ij}) matrices for $q \leq 32$

5	(2,3)
(2)	2

7	(2,3)
(2)	4
(3)	4

8	(2,3)
(2)	5

9	(2,3)	(3,5)
(2)	6	0
(3)	4	2

11	(2,3)	(3,4)
(2)	8	0
(3)	6	2

13	(2,3)	(2,6)	(3,4)
(2)	8	2	0
(3)	4	4	2
(4)	6	0	4

16	(2,3)	(2,5)	(2,6)	(6,7)
(2)	4	1	8	0
(4)	1	4	8	0
(6)	0	0	12	1

Table 1 (continued)

17	(2,3)	(2,5)	(2,6)	(3,7)
(2)	8	2	4	0
(3)	4	0	4	6
(4)	2	4	4	4

19	(2,3)	(2,5)	(3,4)	(3,8)	(4,5)
(2)	8	8	0	0	0
(3)	4	4	6	2	0
(4)	2	8	4	0	2
(8)	0	12	0	4	0

23	(2,3)	(2,5)	(2,6)	(3,4)	(3,7)	(3,14)
(2)	8	4	8	0	0	0
(3)	4	0	4	4	6	2
(4)	2	4	4	6	0	4
(5)	0	4	8	0	4	4

25	(2,3)	(2,5)	(2,6)	(2,10)	(5,7)	(5,8)	(6,7)	(7,8)
(2)	2	8	8	4	0	0	0	0
(5)	0	4	4	4	6	4	0	0
(6)	0	4	4	4	0	4	6	0
(7)	0	8	4	0	4	0	4	2
(8)	0	0	12	0	0	6	0	4

27	(2,3)	(3,4)	(3,5)	(3,7)	(3,10)	(3,15)	(4,6)	(4,11)
(2)	24	0	0	0	0	0	0	0
(3)	4	6	4	4	4	2	0	0
(4)	4	4	0	0	4	4	6	2
(5)	4	0	4	0	8	4	0	4
(11)	4	0	2	6	4	0	4	4

29	(2,3)	(2,5)	(2,6)	(2,13)	(3,4)	(3,7)	(3,11)	(3,13)	(4,9)	(5,6)
(2)	8	8	8	2	0	0	0	0	0	0
(3)	4	0	4	0	4	8	2	4	0	0
(4)	2	8	0	0	4	4	4	0	4	0
(5)	0	4	8	0	0	4	4	2	2	2
(9)	0	4	4	4	2	4	0	4	4	0

Table 1 (continued)

31	(2,3)	(2,5)	(2,6)	(2,8)	(3,4)	(3,7)	(3,8)	(3,10)	(4,9)	(5,6)	(12,13)
(2)	8	8	8	4	0	0	0	0	0	0	0
(3)	4	0	4	0	4	8	4	4	0	0	0
(4)	2	4	0	4	4	0	2	8	4	0	0
(5)	0	4	4	4	2	8	0	0	4	2	0
(6)	0	0	12	0	0	0	0	12	0	4	0
(12)	0	8	4	0	0	4	4	4	2	0	2

32	(2,3)	(2,5)	(2,6)	(2,8)	(2,9)	(2,11)	(6,7)	(4,5)	(4,17)	(14,15)	(16,17)
(2)	4	4	8	4	4	4	1	0	0	0	0
(4)	1	4	0	8	4	4	0	4	4	0	0
(6)	0	0	4	4	4	4	4	0	8	1	0
(14)	0	8	4	4	4	0	0	0	4	4	1
(16)	0	4	4	0	4	8	0	1	4	0	4

It is readily checked that a proper submatrix, which consists of whole columns and has the equal sums in all the rows, does exist only for $q = 17$ and $q = 32$. Three such submatrices, corresponding to the pairs $(17, 4)$, $(32, 4)$ and $(32, 5)$, are shown in Table 2 (the submatrices consist of the rounded columns). The submatrix composed of all the columns of the last two submatrices leads to the pair $(32, 9)$. The remaining four designs mentioned in the theorem correspond to the submatrices, which are complementary w.r.t. the first four. □

Table 2. 4-designs recognized within (λ_{ij}) matrices

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      |-|                            |-|
8  2 |4| 0              4  4  8  4 |4| 4  1  0  0  0  0
4  0 |4| 6              1  4  0  8 |4| 4  0  4  4  0  0
2  4 |4| 4              0  0  4  4 |4| 4  4  0  8  1  0
      |-|               0  8  4  4 |4| 0  0  0  4  4  1
                        0  4  4  0 |4| 8  0  1  4  0  4
 4-(18,5,4)                          |-|
                                               4-(33,5,4)

|-|                |----|   |----|
|4| 4  8  4  4  4 |1  0| 0 |0  0|
|1| 4  0  8  4  4 |0  4| 4 |0  0|
|0| 0  4  4  4  4 |4  0| 8 |1  0|
|0| 8  4  4  4  0 |0  0| 4 |4  1|
|0| 4  4  0  4  8 |0  1| 4 |0  4|     4-(33,5,5)
|-|                |----|   |----|
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3. SOME OBSERVATIONS ON THE CONSTRUCTED 4-DESIGNS

The 4-$(18,5,4)$ design was primarily constructed by W. O. Alltop by a somewhat more specific method, which also uses the orbits of 4-subsets and 5-subsets by action of $PGL(2,q)$. This construction was described in [**3**], Example 8.5, pp. 186–187. It might be said that the above described method, which is based the use of (λ_{ij}) matrices, is an improvement of the Alltop's method. This improvement enables a complete search for 4-designs which arise by action of $PGL(2,q)$.

The constructed 4-$(33,5,5)$ design is recognized as the first member of an infinite class (also constructed by W. O. Alltop in [**2**]) of 4-$(q+1,5,5)$ designs, where q is of the form 2^k for odd numbers $k \geq 5$. More precisely, the families of blocks of designs in this class consist of those orbits by action of $PGL(2,q)$, which have a representative of the form $\{0,1,a,a+1,\infty\}$, for some $a \in \{2,3,...,q-1\}$. The given representatives of the 1., 7., 8., 10. and 11. orbit for $q=32$ in Table 1. are exactly of this form and it is easy to verify that there are no such 5-blocks in other orbits.

The constructed 4-$(33,5,4)$ design is only one orbit of 5-sets by action of $PGL(2,32)$. It is stated in [**4**] that a design with the same parameters arises by action of the 4-homogeneous group $PGamaL(2,32)$, which acts block transitively in such a way that *each* orbit of 5-sets is a 4-design.

Suppose that a 4-$(q+1,5,\lambda_1)$ design is a union U_1 of orbits of 5-sets. If another 4-$(q+1,5,\lambda_2)$ design is a union U_2, which is disjoint with U_1, then it immediately follows that the sum U_1+U_2 is a 4-$(q+1,5,\lambda_1+\lambda_2)$ design. In this way is obtained the 4-$(33,5,9)$ design.

Finally, if a 4-$(q+1,5,\lambda)$ design corresponds to a union of orbits of 5-sets, then the union of complementary orbits corresponds to a 4-$(q+1,5,q-3-\lambda)$ design. One might say that this second design is obtained by "substracting" the first one from the trivial 4-$(q+1,5,q-3)$ design. This kind of substraction (complementation) does not depend on orbits by action of a group; it can be applied directly to the blocks of arbitrary design. The number of constructed 4-designs is doubled (from 4 to 8) by applying this operation.

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Institute of Mathematics,
University of Novi Sad,
Trg D. Obradovića 4, 21000 Novi Sad,
Yugoslavia

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