Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 5 (1994), 9–12.

A LOWER BOUND OF JENSEN'S FUNCTIONAL

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In this paper we give some estimates of the Jensen's functional:

$$J(f)=\int_0^{2\pi}\log|f(e^{i\theta}|\,\frac{\mathrm{d}\theta}{2\pi}\,,$$

if the function $f\in H^\infty$ and has a concentration at low degrees.

Let $f(z)=\sum_{j\geq 0}a_jz^j$, with $||f||_\infty=\sup_\theta|f(e^{i\theta})|<+\infty$ be a function in H^∞. Let $k\in\mathbf{N}$, $0<d\leq 1$. We say that f has concentration d at degree k if

$$\sum_{j\leq k}|a_j|\geq d||f||_\infty. \tag{1}$$

Other ways of measuring a concentration can be expressed. For instance, the polynomial $f(z)$ has a concentration d, $0<d\leq 1$, at low degrees k, measured by lp-norm ($p\geq 1$) if

$$\Big\{\sum_{j\leq k}|a_j|^p\Big\}^{1/p}\geq d\Big\{\sum_{j\geq 0}|a_j|^p\Big\}^{1/p} \tag{2}$$

(for detail see [**3**], [**9**], [**10**]).

This concept was introduced by P. Enflo and B. Beauzamy in [**1**] and [**7**], where it was used in order to obtain, for products of polynomials, estimates from below independent of the degrees. Otherwise, this nation plays an important role in the construction of the operator on a Banach space with no non-trivial subspace (see [**7**]).

[0]1991 Mathematics Subject Classification: 30C10

In [**4**] B. Beauzamy showed that there exists a constant $\hat{C}(d,k)$, depending only on d and k, such that for any $f \in H^\infty$, $||f||_\infty \le 1$, satisfying

$$\left\{ \sum_{j \le k} |a_i|^2 \right\}^{1/2} \ge d \tag{3}$$

it is true that:

$$J(f) = \int_0^{2\pi} \log|f(e^{i\theta})| \frac{\mathrm{d}\theta}{2\pi} \ge \hat{C}(d,k). \tag{4}$$

For our purpose here, $C(d,k)$ will denote the largest such constant possible in (4), i.e.

$$C(d,k) := \inf \Big\{ J(f) : f \in H^\infty, \ ||f||_\infty \le 1 \text{ and satisfying (3)} \Big\}. \tag{5}$$

From [**4**] it follows that $C(d,k) \ge \sup_{t>1} f_{d,k}(t)$, where

$$f_{d,k}(t) = t \log d \left(\frac{t-1}{t+1} \right)^k .$$

We observe that for $k = 0$, $C(d,0) \ge \sup_{t>1} t \log t = \log d$: that is just the classical Jensen's inequality. Otherwise, the precise value of $C(d,k)$, $k > 0$ is unknown. However, B. Beauzamy showed in [**3**] that $C(d,1)$ is the unique number $c < 0$, solution of the equation $e^c(1-2c) = d$.

In the sequel, we have restricted ourselves to the numerical asymptotic estimates for $C(d,k)$, when $k \to +\infty$.

Theorem 1. *If $0 < d < 1$, then $C(d,k) \ge -2k$ asymptotically, when $k \to +\infty$.*

Proof. Firstly, we represent $f_{d,k}(t)$ in the form:

$$f_{d,k}(t) = t \log d + tk \log(t-1) - tk \log(t+1). \tag{6}$$

Now, we observe that $\lim_{t \to 1+} f_{d,k}(t) = -\infty$ and $\lim_{t \to +\infty} f_{d,k}(t) = -\infty$; therefore a maximum exists. We compute the derivatives:

$$f_{d,k}'(t) = \log d + k \log(t-1) - k \log(t+1) + \frac{tk}{t-1} - \frac{tk}{t+1}, \tag{7}$$

$$f_{d,k}''(t) = \frac{-4k(t+1)}{(t^2-1)^2} < 0. \tag{8}$$

We also observe that $\lim_{t \to 1+} f_{d,k}'(t) = +\infty$, $\lim_{t \to +\infty} f_{d,k}'(t) = \log d$, that is $f_{d,k}'(t)$ decreases, because $f_{d,k}''(t) < 0$. Since $\log d < 0$, it follows that there exists exactly one $t_k > 1$ such that $f_{d,k}'(t_k) = 0$, that is

$$C(d,k) \ge \max_{t>1} f_{d,k}(t) = f_{d,k}(t_k).$$

From the equality $f_{d,k}^{'}(t) = 0$ we get with $t = t_k$

$$k = \frac{(t^2-1)\log d}{-2t + (t^2-1)\log(t+1) - (t^2-1)\log(t-1)} \tag{9}$$

from which we easily deduce that $t_k \to +\infty$ if and only if $k \to +\infty$.

Writing $\log(t \pm 1) = \log t + \log(1 \pm 1/t)$ and substituting TAYLOR expansion of order 3 for $\log(1 \pm 1/t)$ when $k \to +\infty$, we get

$$k \approx -\frac{3}{4}\, {t_k}^3 \log d. \tag{10}$$

All we have to do now is to compute $f_{d,k}(t_k)$, and this follows easily by the estimate (10) and the equality for $\log(t \pm 1)$.

Hence, this proves that $f_{d,k}(t_k) \approx -2k$, $k \to +\infty$, that is to asymptotic estimates: $C(d,k) \geq -2k$, when $k \to +\infty$ and $0 < d < 1$.

REMARK 1. The above result can be compared with [**3**], Theorem 1 as and (9), Proposition 2 and [**10**], Theorems 4 and 5.

Finally, we shall consider the case $d = 1$, that is, if the function $f(z)$ (resp. polynomial) has a concentration $d = 1$ at degree k. We first observe that if $d = 1$ and f satisfies (2), then $f(z)$ has degree k, i.e. $f(z) = \sum_{j=0}^{k} a_j z^j$.

From the following example it follows that this is not true if $d = 1$ and f satisfies (1).

EXAMPLE 1. Let $f(z) = 1 - 2z - z^2 - z^3$. We compute that $||f||_\infty = 3$, $\sum_{j \leq 1} |a_j| = 3$. It is clear that $f(z)$ has concentration $d = 1$ at degree 1 in sense (1), but $f(z) \not\equiv 1 - 2z$. Hence we can investigate the largest constant $C(1,k)$. From [**3**] it follows that $C(1,1)$ is the unique solution $(c < 0)$ of the equation $e^c(1 - 2c) = 1$.

For the constant $C(1,k)$ we have the following:

Theorem 2. *$C(1,k) \geq -2k$ for every $k \in \mathbf{N}$.*

Proof. Since $d = 1$, we have that, when $t \to +\infty$,

$$\begin{aligned} f_{1,k}(t) &= tk\log(t-1) - tk\log(t+1) = tk\left(-\frac{2}{t} - \frac{2}{3t^3} + o\left(\frac{1}{t^3}\right)\right) \\ &= = -2k - \frac{2k}{3t^2} + o\left(\frac{1}{t^2}\right) < -2k \end{aligned}$$

and $\dfrac{1}{k} f_{1,k}^{'}(t) = \dfrac{4}{3t^3} + o\left(\dfrac{1}{t^3}\right)$ i.e. $f_{1,k}^{'}(t) > 0$.

Hence, $\sup_{t>1} f_{1,k}(t) = -2k$, that is $C(1,k) \geq -2k$.

This proves the theorem.

REMARK 2. The precise value of $C(1,k)$ is also unknown. But, if $f(z) = \sum_{j \geq 0} a_j z^j$, $|f|_1 = 1$, is an analytic function with concentration $d = 1$ at degree k, measured by l_1-norm, then:

$$C(1,k) = -k \log 2 \tag{11}$$

(see [**11**] for $d \neq 1$ and HURWITZ polynomials).

Indeed, from [**8**] and the fact that $f(z)$ has exactly degree k, that is $f(z) = \sum_{j=0}^{k} a_j z^j$, it follows that

$$-k \log 2 \leq J(f) = \int_0^{2\pi} \log |f(e^{i\theta})| \frac{\mathrm{d}\theta}{2\pi} \leq 0.$$

Since $\int_0^{2\pi} \log |p(e^{i\theta})| \frac{\mathrm{d}\theta}{2\pi} = -k \log 2$, where $p(z) = \frac{1}{2^k}(z+1)^k$ is the extremal polynomial, (11) follows.

We also observe that

$\sup\left\{ J(f) \ : \ ||f||_\infty \leq 1 \text{ and } f \text{ satisfies (3)} \right\} = 0$, where $p(z) \equiv 1$ is the extremal polynomials.

REFERENCES

1. B. BEAUZAMY, P. ENFLO: *Estimations de produits de polynômes.* Journal of Number Theory **21**, 3 (1985), 390–412.
2. B. BEAUZAMY: *Jensen's inequality for polynomials with concentration at low degrees.* Numer. Math. **49** (1986), 221–225.
3. B. BEAYZAMY: *A minimization problem connected with generalized Jensen's inequality.* Math. Anal. and Appl. **145**, 1 (1990), 137–144.
4. B. BEAUZAMY: *Estimations for H^2 functions with concentration at low degrees and applications to complex symbolic computation. To appear.*
5. S. CHON: *On the roots of polynomials with concentration at low degrees.* Math. Anal. and Appl. **149**, 2 (1990), 424–436.
6. S. CHON: *Séries de Taylor et concentration aux bas degrés.* Thèse, Université de Paris VI, 1990.
7. P. ENFLO: *On the invariant subspace problem in Banach spaces.* Acta Math. **158** (1987), 213–313.
8. K. MAHLER: *An application of Jensen's formula to polynomials.* Mathematika **7** (1960), 98–100.
9. S. RADENOVIĆ: *Some estimates of the integral* $\int_0^{2\pi} \log |P(e^{i\theta})| \frac{\mathrm{d}\theta}{2\pi}$. Publ. Inst. Math. (Beograd) **52** (66) (1992), 37–42.
10. S. RADENOVIĆ: *A note on Jensen's functional.* Math. Balkanica **7** (1993), to appear.
11. A. RINGLER, R. S. TRIMBLE, R. S. VARGA: *Sharp lower bounds for a generalized Jensen inequality.* Rocky Mountain J. of Math. To appear.

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(Received January 18, 1993)