UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. 5 (1994), 3–7

A NOTE ON SCHWARZ'S INEQUALITY

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Some improvement of the well-known Schwarz inequality in inner product spaces are given.

1. Let X be a linear space over the real or complex number field $\mathbf{K}$. A mapping $(\ ,\ ) : X \times X \to \mathbf{K}$ will be called positive hermitian form if the following conditions are satisfied:

(i) $(\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z)$ for all x, y, z in X and α, β in $\mathbf{K}$;

(ii) $(y,x) = \overline{(x,y)}$ for all x, y in X;

(iii) $(x,x) \geq 0$ for all x in X.

It is well-known that, if $(\ ,\ )$ is as above and $||x|| := [(x,x)]^{1/2}$, $x \in X$, denotes the semi-norm associated to this norm, then the following inequality (called in literature as the SCHWARZ inequality)

$$||x||\,||y|| \geq |(x,y)| \qquad \text{for all } x, y \text{ in } X \tag{1}$$

holds.

The main purpose of this note is to improve this result as in the sequel.

2. We start to the following result:

Theorem. *Let $(\ ,\ )$ be a positive hermitian form and $\Big\{(\ ,\ )_i\Big\}_{i\in I}$ a family of such functionals so that:*

$$||x||^2 \geq \int_{i\in I} ||x||_i^2 \quad \textit{for all } x \textit{ in } X. \tag{2}$$

Then one has the inequality :

[0]1991 Mathematics Subject Classification: 26D15

$$(3)\qquad ||x||\,||y|| - |(x,y)| \geq \int_{i\in I} \Big[||x||_i||y||_i - |(x,y)_i|\Big] \geq 0$$

for all x, y in X.

Proof. Let H be a finite part of I. If we put:

$$(x,y)_H := (x,y) - \sum_{i\in H}(x,y)_i, \quad x,y \in X,$$

then we observe, by (2), that $(\ ,\ )_H$ is also a positive hermitian form. Applying SCHWARZ inequality (1) for $(\ ,\ )_H$, we deduce that:

$$(4)\qquad \Big(||x||^2 - \sum_{i\in H}||x||_i{}^2\Big)\Big(||y||^2 - \sum_{i\in H}||y||_i{}^2\Big) \geq \Big|(x,y) - \sum_{i\in H}(x,y)_i\Big|^2,$$

for all x, y in x.

On the other hand, ACZÉL's inequality [**6**] yields that

$$(5)\qquad \Big(||x||\,||y|| - \sum_{i\in H}||x||_i||y||_i\Big)^2 \geq \Big(||x||^2 - \sum_{i\in H}||x||_i{}^2\Big)\Big(||y||^2 - \sum_{i\in H}||y||_i{}^2\Big)$$

for all x, y in X, and since

$$||x||\,||y|| \geq \Big(\sum_{i\in H}||x||_i{}^2\sum_{i\in H}||y||_i{}^2\Big)^{1/2} \geq \sum_{i\in H}||x||_i||y||_i$$

hence from (4) and (5) we get:

$$\begin{aligned} ||x||\,||y|| - \sum_{i\in H}||x||_i||y||_i &= \Big|||x||\,||y|| - \sum_{i\in H}||x||_i||y||_i\Big| \\ \geq \Big|(x,y) - \sum_{i\in H}(x,y)_i\Big| &\geq \Big||(x,y)| - \Big|\sum_{i\in H}(x,y)_i\Big|\Big| \\ \geq |(x,y)| - \Big|\sum_{i\in H}(x,y)_i\Big| &\geq |(x,y)| - \sum_{i\in H}|(x,y)_i|, \end{aligned}$$

which implies that:

$$||x||\,||y|| - |(x,y)| \geq \sum_{i\in H}\Big[||x||_i||y||_i - |(x,y)_i|\Big]$$

for every x, y in X and for all H a finite part of I. Consequently, the family $\{||x||_i||y||_i - |(x,y)_i|\}_{i\in I}$ is sumable in $\mathbf{K}$ and the inequality (2) holds.

Corollary 1. *In the above assumptions, we also have the following refinement of the triangle inequality:*

$$(6) \qquad \Big(||x||+||y||\Big)^2 - ||x+y|| \geq \int\limits_{i\in I} \left(\Big(||x||_i+||y||_i\Big)^2 - ||x+y||_i{}^2\right) \geq 0$$

for every x, y two elements in X.

Proof. By a similar argument to that used in the above theorem, we have the inequality:

$$||x||\,||y|| - \mathrm{Re}\,(x,y) \geq \int\limits_{i\in I} \Big(||x||_i||y||_i - \mathrm{Re}\,(x,y)_i\Big) \geq 0$$

for all x, y in X. Since

$$\big(||x||+||y||\big)^2 - ||x+y||^2 = 2\Big(||x||\,||y|| - \mathrm{Re}\,(x,y)\Big)$$

and

$$\big(||x||_i+||y||_i\big)^2 - ||x+y||_i{}^2 = 2\Big(||x||_i||y||_i - \mathrm{Re}\,(x,y)_i\Big)$$

for all x, y in X, the inequality (6) is thus proven.

Corollary 2. *Let $(\ ,\ )_1$ and $(\ ,\ )_2$ be two positive hermitian forms with $||\cdot||_2$ is greater than $||\cdot||_1$, i.e., $||x||_2 \geq ||x||_1$ for all x in X. Then:*

$$||x||_2||y||_2 - |(x,y)_2| \geq ||x||_1||y||_1 - |(x,y)_1| \geq 0$$

for all x, y in X.

Corollary 3. *Let $||\cdot||_2$ and $||\cdot||_1$ be as above. Then*

$$\big(||x||_2+||y||_2\big)^2 - ||x+y||_2{}^2 \geq \big(||x||_1+||y||_1\big)^2 - ||x+y||_1{}^2 \geq 0,$$

for all x, y in X.

Further on, we will give some natural applications of these results.

Applications. a. Let $(X; (\ ,\ ))$ be an inner product space and $A : X \to X$ is a bounded linear operator on X. Denote $||A|| := \sup\big\{||Ax||, ||x|| \leq 1\big\}$. Then the inequaility:

$$||A||^2\big(||x||\,||y|| - |(x,y)|\big) \geq ||Ax||\,||Ay|| - |(Ax,Ay)| \geq 0$$

holds for all x, y in X.

The proof follows by Corollary 2 for $(x,y)_2 := ||A||^2(x,y)$ and $(x,y)_1 := (Ax,Ay),\ x,y \in X$.

b. Let $A : X \to X$ be a linear operator on X so that $||Ax|| \geq \lambda||x||$ for all x in X $(\lambda > 0)$. Then:

$$||Ax||\,||Ay|| - |(Ax,Ay)| \geq \lambda^2\big(||x||\,||y|| - |(x,y)|\big) \geq 0$$

for all x, y in X.

The proof is obvious from Corollary 2 for $(x,y)_2 := (Ax, Ay)$ and $(x,y)_1 := \lambda^2(x,y)$ with $x, y \in X$.

c. Suppose that $A : X \to X$ is symmetric, i.e. $(Ax, y) = (y, Ax)$ for all x, y in X and positive definite with the constant $m > 0$, i.e., $(Ax, x) \geq m||x||^2$ for all x in X. Then the inequality:

$$(Ax,x)^{1/2}(Ay,y)^{1/2} - |(Ax,y)| \geq m\big[||x||\,||y|| - |(x,y)|\big] \geq 0$$

holds for every x, y in X.

The proof follows by Corollary 2 for the positive hermitian forms: $(x,y)_2 := (Ax, y)$ and $(x,y)_1 := m\,(x,y),\ x, y \in X$ and we will omit the details.

d. Suppose that $(e_i)_{i\in I}$ is an orthogonal family of vectors in the inner product space $(X;(\ ,\ ))$, i.e., $(e_i, e_j) = \delta_{ij},\ \ i, j \in I$. Then one has the following refinement of SCHWARZ inequality:

$$||x||\,||y|| - |(x,y)| \geq \left(\int_{i\in I} |(x,e_i)|^2\right)^{1/2}\left(\int_{i\in I} |(y,e_i)|^2\right)^{1/2} - \left|\int_{i\in I}(x,e_i)(e_i,y)\right| \geq 0$$

for all x, y in X.

The argument follows by Corollary 2 for the positive hermitian forms: $(x,y)_2 := (x,y)$ and $(x,y)_1 := \int\limits_{i\in I}(x,e_i)(e_i,y)$, where x, y are in X. The fact that $||\cdot||_2$ is greater than $||\cdot||_1$ follows by BESSEL's inequality [**5**]:

$$||x||^2 \geq \int_{i\in I} |(x,e_i)|^2,$$

which is valid for all x in X.

e. Let x_i, y_i be complex numbers $(i = 1, \ldots, n)$ and $p_i \geq q_i \geq 0$ for all $i = 1, \ldots, n$. Then we have the following inequalities:

$$\begin{aligned} &\left(\sum_{i=1}^n p_i|x_i|^2\right)^{1/2}\left(\sum_{i=1}^n p_i|y_i|^2\right)^{1/2} - \left|\sum_{i=1}^n p_i x_i y_i\right| \\ \geq\ &\left(\sum_{i=1}^n q_i|x_i|^2\right)^{1/2}\left(\sum_{i=1}^n q_i|y_i|^2\right)^{1/2} - \left|\sum_{i=1}^n q_i x_i y_i\right| \geq 0. \end{aligned}$$

f. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbf{R}\cup\{\infty\}$. Denote $L^2(\Omega, w)$ the HILBERT space of all functions x with complex values which are defined on and are 2-w-integrable on Ω, i.e., $\int\limits_{\Omega} w(s)|x(s)|^2 \mathrm{d}\mu(s) < \infty$, where w is a positive measurable function on Ω.

If $w \geq v \geq 0$ and $x, y \in L^2(\Omega, w)$ (v is also a measurable function on Ω), then we have the following inequalities:

$$\left(\int_\Omega w(s)|x(s)|^2\,\mathrm{d}\mu(s)\right)^{1/2}\left(\int_\Omega w(s)|y(s)|^2\,\mathrm{d}\mu(s)\right)^{1/2} - \left|\int_\Omega w(s)x(s)y(s)\,\mathrm{d}\mu(s)\right|$$

$$\geq \left(\int_\Omega v(s)|x(s)|^2\,\mathrm{d}\mu(s)\right)^{1/2}\left(\int_\Omega v(s)|y(s)|^2\,\mathrm{d}\mu(s)\right)^{1/2} - \left|\int_\Omega v(s)x(s)y(s)\,\mathrm{d}\mu(s)\right| \geq 0.$$

For other inequalities connected with SCHWARZ's result in inner product spaces we refer to [**1**–**3**], where further references are given.

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(Received September 15, 1992)