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ON IRREGULAR SAMPLING THEOREMS FOR FUNCTIONS BANDLIMITED IN A GENERALIZED SENSE

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In this work we give two new sampling theorems which can be thought as extension of Seip's result [1]. First theorem allow us to reconstruct function from a class which is broader than Paley - Wiener class of functions. The second theorem extends Seip's sampling expansion to functions of two or more variables.

1. INTRODUCTION

We will consider reconstruction of some class of entire functions based on their values taken at the isolated sampling points. The function f is called an entire function if it is homomorphic in the whole complex plane $\mathcal{C}$.

Let $M_f(r)$ denote the function $M_f(r) = \max_{|z|\leq r} |f(z)|$, which is an increasing function of r. The numbers defined by

$$(1) \qquad \rho = \overline{\lim_{r\to\infty}} \frac{\ln\ln M_f(r)}{\ln r}, \quad \sigma = \overline{\lim_{r\to\infty}} \frac{\ln M_f(r)}{r^\rho}$$

are called order and type of an entire function, respectively. If $\rho = 1$ and $0 < \sigma < \infty$, the corresponding function f is called an entire function of exponential type σ.

An entire function of exponential type σ belongs to the so-called Wiener class W_σ if $\int_{-\infty}^{\infty} |f(\lambda)|^2 |d\lambda < \infty$.

From the Paley-Wiener Theorem [**2**], the parameterization of functions from the class W_σ is as follows:

$$(2) \qquad W_\sigma = \left\{ \int_{-\sigma}^{\sigma} e^{i\lambda t} g(t)\,\mathrm{d}t, \;\; g \in L^2(-\sigma,\sigma) \right\}.$$

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It is well known that the classical SHANNON - KOTELNIKOF sampling theorem may be generalized in the following way [**1**].

Theorem 1. *Let* $\{t_n\}$, $n \in \mathcal{Z}$ *be a sequence of real numbers such that*

$$q = \sup_{n\in\mathcal{Z}} |t_n - n| < 1/4 \ . \tag{3}$$

Then for all $f \in W_\sigma$, $\sigma < \pi$,

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{G(t)}{G'(t_n)(t-t_n)} \ , \tag{4}$$

uniformly on all compact subset of $\mathcal{C}$, *where*

$$G(t) = (t - t_0) \prod_{i=1}^{\infty} \left(1 - \frac{t}{t_n}\right) \left(1 - \frac{t}{t_{-n}}\right) . \tag{5}$$

It is easy to see that for $t_n = n$, $n \in \mathcal{Z}$, $G(t) = \sin \pi t$, and (4) becomes

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{\sin \pi t}{\pi} (-1)^n \frac{f(n)}{t-n} \ , \tag{6}$$

which is the classical SHANNON - KOTELNIKOF sampling theorem.

GENERALIZATION OF THEOREM 1

Our first generalization of Theorem 1 is toward extending the class of functions f from the PALEY - WIENER class to a more general one.

Theorem 2. *Let* $\{t_n\}$ *be a sequence, such that* $t_0 = 0$, $t_n = t_{-n}$, $n = 1, 2, \cdot$, $\sup |t_n - n| \le q < 1/10$. *Then for all entire functions* f *of exponential type with indicator diagram of width equal to* $2k < 2\pi$, *and* $\sup_{n\in\mathcal{Z}} |f(t_n)| < \infty$,

$$f(t) = -\sum_{n=-\infty}^{\infty} \frac{f(t_n)}{\omega H'(t_n)} \ \frac{H(t)H(\omega(t-t_n))}{(t-t_n)^2} \ , \tag{7}$$

which converges uniformly on each compact subset of $\mathcal{C}$, *where* ω *is any fixed number in the range* $0 < \omega < 1 - \frac{k}{\pi}$. *The function* $H(t)$ *has the form*

$$H(t) = t \prod_{n=1}^{\infty} \left(1 - \frac{t^2}{{t_n}^2}\right) . \tag{8}$$

Proof. Without loss of generality we assume that growth indicator of f satisfies

$$h_f\left(\frac{\pi}{2}\right) = h_f\left(-\frac{\pi}{2}\right) = k < \pi. \tag{9}$$

Let $0 < \omega < 1 - \frac{k}{\pi}$, and t be a fixed complex number. Consider the function

$$g(\xi) = f(\xi)\, \frac{H(\omega(\xi - t))}{\omega(\xi - t)}\, \frac{1}{H(\xi)}\ , \tag{10}$$

which is meromorphic with simple poles t_n . Note that residuals of g has the form

$$\mathrm{Res}_{\xi = t_n} g(\xi) = f(t_n)\, \frac{H(\omega(t_n - t))}{\omega(t_n - t)}\, \frac{1}{H'(t_n)}\ . \tag{11}$$

From assumption $\sup_{n\in\mathcal{N}} |t_n - t| = q < 1/10$, and asymptotic properties of Gamma functions, we obtain

$$|H'(t_n)| \geq K_0(1 + |n|)^{-6q}, \tag{12}$$

where constant $K_0 > 0$ is independent of n.

Consider now the series

$$\sum_{n=-\infty}^{\infty} f(t_n)\, \frac{H(\omega(t_n - t))}{\omega(t_n - t)}\, \frac{1}{H'(t_n)}\, \frac{1}{\xi - t_n}\ . \tag{13}$$

Since

$$|H(\xi)| \leq A_0(1 + |\xi|)^{4q} e^{\pi |Im \xi|}, \tag{14}$$

where A_0 is a constant independent of ξ, see [**3**], from (12) and assumption of Theorem 2 we conclude that series (13) is uniformly convergent on each compact subset of $\mathcal{C}$ which does not include points t_n . From this fact stems that function

$$R(\xi) = g(\xi) - \sum_{n=-\infty}^{\infty} f(t_n)\, \frac{H(\omega(t_n - t))}{\omega(t_n - t)}\, \frac{1}{H'(t_n)}\, \frac{1}{\xi - t_n}\ , \tag{15}$$

is an entire function of exponential type. It is obvious that

$$h_R\left(\pm \frac{\pi}{2}\right) = k + \omega\pi - \pi < 0,$$

and taking into consideration that indicator function $h_R(\theta)$ is continous we have for angles near $\pm\pi/2$ i.e. $|\theta \pm \frac{\pi}{2}| < \epsilon$, that $h_R(\theta)$ is negative. From this fact and the Phragman - Lindelofe theorem [**1**] we conclude that $R(\xi) = \mathrm{const}$. Moreover, when $|\xi| \to \infty$, $R(\xi) \to 0$, which implies that $R(\xi) \equiv 0$, and from (10) and (15)

$$f(\xi)\, \frac{H(\omega(\xi - t))}{\omega(\xi - t)}\, \frac{1}{H(\xi)} = \sum_{n=-\infty}^{\infty} \frac{f(t_n)}{\omega(t_n - t)}\, \frac{H(\omega(t_n - t))}{H'(t_n)}\, \frac{1}{\xi - t_n}. \tag{16}$$

When $\xi \to t$, from (16) we obtain

$$f(t) = -\sum_{n=-\infty}^{\infty} \frac{f(t_n)}{\omega H'(t_n)}\, \frac{H(t) H(\omega(t_n - t))}{(t - t_n)^2}, \tag{17}$$

and conclude the proof. □

Remark 1. If $t_n = n$, then $H(t) = \frac{\sin \pi t}{\pi}$, and according to the Theorem 2

$$f(t) = \sum_{n=-\infty}^{\infty} (-1)^{n+1} \frac{f(n)}{\omega 2\pi} \frac{\sin \pi t \sin \pi(\omega(n-t))}{(t-n)^2} , \tag{18}$$

which is the well - known self-truncating expansion of HELMS and THOMAS [**4**].

Remark 2. One can applied Theorem 2 to the functions which do not belonge to PALEY - WIENER class of functions. For example if $f(z) = \int_a^b e^{iz\varphi(t)} h(t)\, \mathrm{d}t$, $h \in L^2(a,b)$, and φ is continous on $[a,b]$ such that

$$\operatorname{conv}\{i\varphi(t) : a \le t \le b\} \subset \{z : |Imz| \le \sigma < \pi\},$$

then f can be reconstructed based on sampling points t_n by the formula (7).

Our second generalization of Theorem 1 is toward reconstruction of function of two or more variables.

Theorem 3. *Let u_n and v_n be two sequences with properties*

$$\sup_{n\in\mathcal{Z}} |u_n - n| \le q < 1/4, \quad \sup_{n\in\mathcal{Z}} |v_n - n| \le q < 1/4.$$

If function of two variables $f(z_1,z_2)$ is representable in the form

$$f(z_1,z_2) = \int_{-\sigma}^{\sigma}\int_{-\sigma}^{\sigma} e^{it_1z_1+it_2z_2}\varphi(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2, \tag{19}$$

$$\varphi \in L^2((-\sigma,\sigma)\times(-\sigma,\sigma)), \quad \sigma_1^2+\sigma_2^2 < \pi, \tag{20}$$

then,

$$f(x_1,x_2) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{G_1(x_1)G_2(x_2)}{G_1'(u_n)G_2'(v_m)} \frac{f(u_n,v_m)}{(x_1-u_n)(x_2-v_m)}, \tag{21}$$

where

$$G_1(t) = (t-u_0)\prod_{i=1}^{\infty}\left(1-\frac{t}{u_n}\right)\left(1-\frac{t}{u_{-n}}\right), \tag{22}$$

$$G_2(t) = (t-v_0)\prod_{i=1}^{\infty}\left(1-\frac{t}{v_n}\right)\left(1-\frac{t}{v_{-n}}\right). \tag{23}$$

Proof. Consider the function

$$f(z_1,z_2) = \int_{-\sigma_2}^{\sigma_2} e^{it_2z}\,\mathrm{d}t_2\left(\int_{-\sigma_1}^{\sigma_1} e^{it_1z_1}\varphi(t_1,t_2)\,\mathrm{d}t_1\right),$$

where $z_1 = x_1 + y_1$ is a fixed complex number. Denote by h the function

$$h(t_2) = \int_{-\sigma_1}^{\sigma_1} e^{it_1 z_1}\, \varphi(t_1, t_2)\, dt_1.$$

From the Cauchy - Buniakovsky inequality follows

$$|h(t_2)|^2 \le \frac{\sinh 2\sigma_1 y_1}{y_1} \int_{-\sigma_1}^{\sigma_1} |\varphi(t_1, t_2)|^2\, \mathrm{d}t_1, \tag{24}$$

and consequently

$$\int_{-\infty}^{\infty} |h(t_2)|^2\, \mathrm{d}t_2 \le \frac{\sinh 2\sigma_1 y_1}{y_1} \int_{-\sigma_1}^{\sigma_1} \int_{-\sigma_2}^{\sigma_2} |\varphi(t_1, t_2)|^2\, \mathrm{d}t_1 \mathrm{d}t_2 < \infty. \tag{25}$$

Therefore we are able to coclude that

$$\int_{-\sigma_1}^{\sigma_1} e^{it_1 u_n} \varphi(t_1, t_2)\, \mathrm{d}t_1 \in L^2(-\sigma_2, \sigma_2), \tag{26}$$

and due to Theorem 1

$$f(u_n, x_2) = \sum_{m=-\infty}^{\infty} \frac{G_2(x_2)}{G_2'(v_m)(x_2 - v_m)}\, f(u_n, v_m). \tag{27}$$

If we apply Theorem 1 again, (27) becomes

$$f(x_1, x_2) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{G_1(x_1) G_2(x_2)}{G_1'(u_n) G_2'(v_m)(x_1 - u_n)(x_2 - v_m)}. \tag{28}$$

Therefore, we conclude proof of Theorem 3. □

Note that following the above reasoning it is easy to generalize Theorem 3 for functions of more than two variables.

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