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ON GRAPHS WHOSE SECOND LARGEST EIGENVALUE DOES NOT EXCEED $\sqrt{2}-1$

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Graphs whose second largest eigenvalue does not exceed $\sqrt{2}-1$ are characterized.

1. INTRODUCTION

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ be the eigenvalues of a graph G on n vertices.

The second largest eigenvalue $\lambda_2 = \lambda_2(G)$ of a graph G appears several times in the literature. Of special importance are graphs for which $\lambda_2(G)$ has a small value.

A. J. Hoffman posed the problem of characterizing graphs with the second largest eigenvalue not greater than 1. An explicit characterization of connected bipartite graphs with the property $\lambda_2(G) \leq 1$ is given in [**4**].

P. Cao and H. Yuan[**1**] determine graphs without isolated vertices with the property $0 < \lambda_2(G) < 1/3$. They also posed the problem of characterizing graphs with the property $1/3 < \lambda_2(G) < (\sqrt{5}-1)/2$.

In this paper all simple graphs without isolated vertices with the property $0 < \lambda_2(G) < \sqrt{2}-1$ are determined. We prove that a graph G without isolated vertices has this property if and only if G belongs to one of three classes of graphs with corresponding values of the respective parameters.

The union $G_1 \cup G_2$ of graphs $G_1 = (V_1, U_1)$ and $G_2 = (V_2, U_2)$ $(V_1 \bigcap V_2 = \emptyset)$ is the graph $G = (V, U)$, for which $V = V_1 \cup V_2$ and $U = U_1 \cup U_2$.

The complete product $G_1 \nabla G_2$ of graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2.

The complete multipartite graph $K_{s_1,\ldots,s_m}$ is the complete product of disjoint graphs $\overline{K}_{s_1}, \ldots, \overline{K}_{s_m}$ i.e., $K_{s_1,\ldots,s_m} = \overline{K}_{s_1} \nabla \cdots \nabla \overline{K}_{s_m}$.

In particular, nG denotes $G \cup G \cup \cdots \cup G$ and $\nabla_n G$ denotes $G \nabla G \nabla \cdots \nabla G$.

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2. LEMMAS

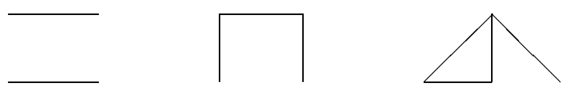
Let V' be a subset of vertices of a graph G and $|V'| = k$. Denote by $G - V'$ the subgraph obtained from G by deleting all vertices in V' together with incident edges. The inequalities in the following lemma are known as Cauchy's inequalities.

Lemma 1 (see, for example, [**3**], p. 19). *For* $1 \le i \le n-k$,

$$\lambda_i(G) \ge \lambda_1(G - V') \ge \lambda_{i+k}(G).$$

Lemma 2 (E. S. Wolk [**6**].) *If* $\overline{G}$ *is a connected graph and* G *has no isolated vertices, then* G *contains an induced subgraph isomorpic to* $2K_2$ *or* P_4.

Lemma 3 (J. H. Smith, [**5**]). *If a simple graph without isolated vertices does not contain as an induced subgraph any of graphs* $2K_2$, P_4 *and* $K_1\nabla(K_1 \cup K_2)$ *in Fig. 1. then* G *is a complete multipartite graph.*

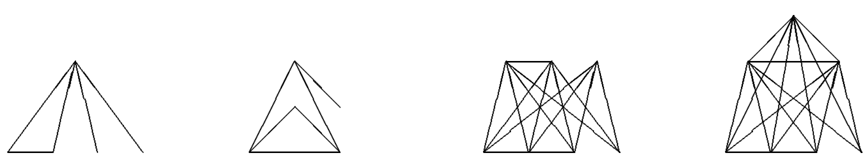


$2K_2$ P_4 $K_1\nabla(K_1 \cup K_2)$

Fig. 1

The following lemma is a consequence of a routine calculation.

Lemma 4. *Let* H_1, H_2, H_3 *and* H_4 *be the graphs as in Fig 2. Then* $\lambda_2(H_i) > \sqrt{2}-1 (1 \le i \le 4)$.



$H_1 = K_1\nabla(K_1 \cup \overline{K}_2)$ $H_2 = K_1\nabla(K_1 \cup K_3)$ $H_3 = (K_1 \cup K_2)\nabla(K_1 \cup P_3)$ $H_4 = (K_1 \cup P_3)\nabla K_3$

Fig. 2

Lemma 5 (D. Cvetković [**2**], [**3**], p. 57). *The characteristic polynomial of the* ∇*-product of graphs* G_1 *and* G_2 ($|G_1| = n_1, |G_2| = n_2$) *is given by the relation*

$$P(G_1\nabla G_2, \lambda) = (-1)^{n_2} P(G_1, \lambda) P(\overline{G}_2, -\lambda-1)$$
$$+(-1)^{n_1} P(G_2, \lambda) P(\overline{G}_1, -\lambda-1)(-1)^{n_1+n_2} P(\overline{G}_1, -\lambda-1) P(\overline{G}_2, -\lambda-1).$$

Lemma 6. *We have*

(1) $$P(\nabla_n(K_1 \cup K_2), \lambda)$$
$$= (\lambda+1)^{n-1}(\lambda^2+2\lambda-1)^{n-1}(\lambda^3 - 3(n-1)\lambda^2 - (2n-1)\lambda + n - 1);$$

(2) $$P((\nabla_n(K_1 \cup K_2))\nabla(\nabla_m \overline{K}_p), \lambda)$$
$$= \lambda^{m(p-1)}(\lambda+p)^{m-1}(\lambda+1)^{n-1}(\lambda^2+2\lambda-1)^{n-1}(\lambda^4 - 3(n-1) + p(m-1))\lambda^3$$

$$-(3p(n+m-1)+2n-1)\lambda^2+(p(2n+m-1)-n+1)\lambda+p(n+m-1));$$

(3) $$\lambda_2(\nabla_n(K_1\cup K_2))\nabla(\nabla_m\overline{K}_p))=\sqrt{2}-1 \text{ for } n>1,$$
$$\lambda_2((K_1\cup K_2)\nabla(\nabla_m\overline{K}_p))<\sqrt{2}-1.$$

Proof. We prove (1) by induction. For $n=1$, equality (1) reads

$$P(K_1\cup K_2,\lambda)=\lambda^3-\lambda=\lambda(\lambda^2-1),$$

and it can be proved by a straightforward calculation.

Assume that (1) is valid for a positive integer n, and prove it for $n+1$. Since

$$P(\overline{\nabla_n(K_1\cup K_2)},-\lambda-1)=P(nP_3,-\lambda-1)=(-1)^n(\lambda+1)^n(\lambda^2+2\lambda-1)^n,$$

we have by Lemma 5

$$\begin{aligned}P(\nabla_{n+1}(K_1\cup K_2),\lambda) &= P((\nabla_n(K_1\cup K_2))\nabla(K_1\cup K_2),\lambda)\\ &= -P(\nabla_n(K_1\cup K_2),\lambda)P(\overline{K_1\cup K_2},-\lambda-1)\\ &+ (-1)^nP(K_1\cup K_2,\lambda)P(\overline{\nabla_n(K_1\cup K_2)},-\lambda-1)\\ &- (-1)^{n+1}P(\overline{\nabla_n(K_1\cup K_2)},-\lambda-1)P(\overline{K_1\cup K_2},-\lambda-1).\end{aligned}$$

Whence, by some calculation, we get relation (1) for the next positive integer n+1.

Relation (2) is easy to prove by Lemmas 5 and 6, relation (1), having in mind that

$$\begin{aligned}P(\nabla_m\overline{K}_p,\lambda) &= \lambda^{m(p-1)}(\lambda+p(1-m))(\lambda+p)^{m-1},\\ P(\overline{\nabla_m\overline{K}_p},-\lambda-1) &= P(mK_p,-\lambda-1)=(-1)^{mp}\lambda^{m(p-1)}(\lambda+p)^m.\end{aligned}$$

Let $f(\lambda)=\lambda^4-(3(n-1)+p(m-1))\lambda^3-(3p(n+m-1)+2n-1)\lambda^2+(p(2n+m-1)-n+1)\lambda+p(n+m-1)$. It is easy to check that $f(\lambda)$ has exactly two positive roots and $f(0)>0$, $f(\sqrt{2}-1)<0$, $f(+\infty)>0$. Therefore the positive roots of $f(\lambda)$ lie in intervals $(0,\sqrt{2}-1)$ and $(\sqrt{2}-1,+\infty)$. By (2) we conclude that $\lambda_2((\nabla_n(K_1\cup K_2))\nabla(\nabla_m\overline{K}_p))=\sqrt{2}-1$ if $n>1$ and $\lambda_2((K_1\cup K_2)\nabla(\nabla_m\overline{K}_p))<\sqrt{2}-1$

Lemma 7. $\lambda_2((K_1\cup K_{r,s})\nabla\overline{K}_q)\le\sqrt{2}-1$ $(r\le s)$ *if and only if one of the conditions* 1.–10. *holds*:

1. $r>1,\ s\ge r,\ q=1;$	6. $r=2,\ s=5,\ 3\le q\le 4;$
2. $r=1,\ s\ge 1,\ q\ge 2;$	7. $r=2,\ 6\le s\le 8,\ q=3;$
3. $r=2,\ s\ge 2,\ q=2;$	8. $r=3,\ s=3,\ 2\le q\le 4;$
4. $r=2,\ 2\le s\le 3,\ q\ge 3;$	9. $r=3,\ 4\le s\le 7,\ q=2;$
5. $r=2,\ s=4,\ 3\le q\le 7;$	10. $r=4,\ s=4,\ q=2.$

Proof. Since

$$\begin{aligned}P(K_1\cup K_{r,s},\lambda) &= \lambda^{r+s-1}(\lambda^2-rs),\\ P(\overline{K}_q,\lambda) &= \lambda^q\\ P(\overline{K_1\cup K_{r,s}},-\lambda-1) &= (-1)^{r+s-1}\lambda^{r+s-2}(\lambda^3+(r+s+1)\lambda^2+rs\lambda-rs),\\ P(K_q,-\lambda-1) &= (-1)^q\lambda^{q-1}(\lambda+q),\end{aligned}$$

we have by Lemma 5

$$P((K_1 \cup K_{r,s})\nabla\overline{K}_q, \lambda) = \lambda^{q+r+s-3}(\lambda^4 - (q+qr+qs+rs)\lambda^2 - 2qrs\lambda + qrs).$$

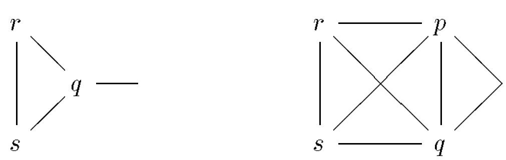
Non-zero eigenvalues of the graph $(K_1 \cup K_{r,s})\nabla\overline{K}_q$ (see Fig. 3) are determined by equation $D(\lambda) = \lambda^4 - (q+qr+qs+rs)\lambda^2 - 2qrs\lambda + qrs = 0$.

The last equation has exactly two positive roots and

$$D(0) = qrs > 0,\quad D(\sqrt{2}-1) = (3-2\sqrt{2})(qrs - rs - qs - qr - q) + (17 - 12\sqrt{2}).$$

Now, it is clear that $\lambda_2((K_1 \cup K_{r,s})\nabla\overline{K}_q) \leq \sqrt{2}-1$ if and only if $D(\sqrt{2}-1) \leq 0$ i.e., $T(q,r,s) = qrs - rs - qs - qr - q < 0$.

This inequality is true if the parameters q,r and s satisfy one of the relations 1.–10.



$(K_1 \cup K_{r,s})\nabla\overline{K}_q$ $(K_1 \cup K_{r,s})\nabla K_{p,q}$

Fig. 3

Lemma 8. $\lambda_2((K_1 \cup K_{r,s})\nabla K_{p,q}) \leq \sqrt{2}-1$ $(r \leq s, p \leq q)$ *if and only if one of the conditions* 1.–5. *holds*:

1. $r=1,\ s=1,\ p\geq 1,\ q\geq p$;
2. $r=1,\ s=2,\ 1\leq p\leq 2,\ q\leq p$;
3. $r=1,\ s=2,\ p=3,\ 3\leq q\leq 7$;
4. $r=1,\ s=2,\ p=4,\ q=4$;
5. $r=1,\ s=3,\ p=1,\ q=1$.

Proof. Since

$$\begin{aligned}
P(K_1 \cup K_{r,s}, \lambda) &= \lambda^{r+s-1}(\lambda^2 - rs),\\
P(K_{p,q}, \lambda) &= \lambda^{p+q-2}(\lambda^2 - pq),\\
P(\overline{K_1 \cup K_{r,s}}, -\lambda-1) &= (-1)^{r+s-1}\lambda^{r+s-2}(\lambda^3 + (r+s+1)\lambda^2 + rs\lambda - rs),\\
P(\overline{K_{p,q}}, -\lambda-1) &= (-1)^{p+q}(\lambda+p)(\lambda+q)\lambda^{p+q-2},
\end{aligned}$$

we have by Lemma 5

$$P((K_1 \cup K_{r,s})\nabla K_{p,q}, \lambda) = \lambda^{p+q+r+s-5}(\lambda^5 - (pq + rs + (p+q)(r+s+1))\lambda^3$$

$$-2(rs(p+q) + pq(r+s+1))\lambda^2 - (3pqrs - rs(p+q))\lambda + 2pqrs).$$

Non-zero eigenvalues of the graph $(K_1 \cup K_{r,s})\nabla K_{p,q}$ (see Fig. 3) are determined by equation

$$D(\lambda) = \lambda^5 - (pq + rs + (p+q)(r+s+1))\lambda^3 - 2(rs(p+q) + pq(r+s+1))\lambda^2$$

$$-(3pqrs - rs(p+q))\lambda + 2pqrs = 0.$$

This equation has exactly two positive roots and $D(0) = -2pqrs < 0$. Hence $\lambda_2((K_1 \cup K_{r,s})\nabla K_{p,q}) \le \sqrt{2}-1$ if and only if $D(\sqrt{2}-1) \ge 0$. It is easy to prove that the last inequality holds if parameters p, q, r and s satisfy one of relations 1.–5.

THE MAIN RESULT

Theorem. *Let G be a graph without isolated vertices. Then $0 < \lambda_2(G) \le \sqrt{2}-1$ if and only if one of the following holds:*

(a) $G = (\nabla_n(K_1 \cup K_2))\nabla K_{s_1,\ldots,s_m}$;

(b) $G = (K_1 \cup K_{r,s})\nabla \overline{K}_q$, *and parameters q, r and s satisfy one of conditions 1.–10. from Lemma7;*

(c) $G = (K_1 \cup K_{r,s})\nabla K_{p,q}$, *and parameters p, q, r and s satisfy one of conditions 1.–5. from Lemma 8.*

Proof. Let $0 < \lambda_2(G) \le \sqrt{2}-1$. Then G must be connected. Otherwise, $2K_2 \subset G$ and, by Lemma 1, $\lambda_2(G) \ge \lambda_2(2K_2) = 1 > \sqrt{2}-1$, a contradiction. The following statements are also true.

1° *$\overline{G}$ is disconnected.* Otherwise, by Lemma 2, G contains $2K_2$ or P_4 as an induced subgraph and, by Lemma 1, $\lambda_2(G) \ge \min\{\lambda_2(2K_2), \lambda_2(P_4)\} = (\sqrt{5}-1)/2 > \sqrt{2}-1$, what is a contradiction.

Hence, $\overline{G} = G_1 \cup G_2 \cup \cdots \cup G_k$ $(k \ge 2)$, where $G_1, G_2, \ldots, G_k$ are components of $\overline{G}$. Note that $\overline{G}_i$ is an induced subgraph of G for each i and

$$G = \overline{G}_1 \nabla \overline{G}_2 \nabla \cdots \overline{G}_k.$$

2° *Each graph $\overline{G}_i$ $(1 \le i \le k)$ contains an isolated vertex.* Otherwise, by Lemma 2, $\overline{G}_i$ contains $2K_2$ or P_4 as an induced subgraph. Thus $\lambda_2(G) \ge \lambda_2(G_i) \ge (\sqrt{5}-1)/2 > \sqrt{2}-1$, a contradiction.

3° *For some i, $\overline{G}_i$ contains the edges.* Otherwise, G is a complete multipartite graph and $\lambda_2(G) \le 0$, a contradiction.

4° *Any graph $\overline{G}_i$ with edges, contains exactly one isolated vertex.* Otherwise, $H_1 \subset G$ and, by Lemmas 1 and 4, $\lambda_2(G) \ge \lambda_2(H_1) > \sqrt{2}-1$, a contradiction. Denote by v_i isolated vertex of a such graph $\overline{G}_i$.

5° *Each graph $\overline{G}_i$, which contains edges, does not contain the graph K_3 as an induced subgraph.* Otherwise, $H_2 \subset G$ and, by Lemmas 1 and 4, $\lambda_2(G) \ge \lambda_2(H_2) > \sqrt{2}-1$, a contradiction.

Thus, $\overline{G}_i$ with edges does not contain circuits of odd length and $\overline{G} - v_i$ is a bipartite graph. It follows that $\overline{G}_i$ does not contain the graph $K_1\nabla(K_1 \cup K_2)$ as an induced subgraph and, by Lemma 3, $\overline{G}_i - v_i$ is a complete bipartite graph.

In the sequal, we distinguish two cases.

Case I. At least two graphs $\overline{G}_i$ contain edges. Then, for any such graph $\overline{G}_i$, $\overline{G}_i = K_1 \cup K_2$. Otherwise, $H_3 \subset G$ and, by Lemmas 1 and 4, $\lambda_2(G) \geq \lambda_2(H_3) > \sqrt{2}-1$, a contradiction.

We conclude that $G = (\nabla_n(K_1 \cup K_2))\nabla K_{s_1,\ldots,s_m}$. Let $p = \max\{s_1, \ldots, s_m\}$. Then $G \subset (\nabla_n(K_1 \cup K_2))\nabla(\nabla_m \overline{K}_p)$ and, by Lemmas 1 and 6,

$$\lambda_2(G) \leq \lambda_2((\nabla_n(K_1 \cup K_2))\nabla(\nabla_m \overline{K}_p)) = \sqrt{2}-1.$$

Case II. Exactly one of the graphs $\overline{G}_i$ contains edges. Let $\overline{G}_1$ has this property.

If $\overline{G}_1 = K_1 \cup K_2$, then $G = (K_1 \cup K_2)\nabla K_{s_1,\ldots,s_m}$ and, by Lemmas 1 and 6, $\lambda_2(G) \leq \lambda_2((K_1 \cup K_2)\nabla(\nabla_m \overline{K}_p)) < \sqrt{2}-1$ $(p = \max\{s_1, \ldots, s_m\})$.

If $\overline{G}_1 = K_1 \cup K_{r,s}$ $(r \leq s,\ s \geq 2)$, then $k \leq 3$. Otherwise, $H_4 \subset G$ and, by Lemmas 1 and 4, $\lambda_2(G) \geq \lambda_2(H_4) > \sqrt{2}-1$, a contradiction. We conclude that in this case $G = (K_1 \cup K_{r,s})\nabla \overline{K}_q$ or $G = (K_1 \cup K_{r,s})\nabla K_{p,q}$. By Lemmas 7 and 8, $\lambda_2(G) \leq \sqrt{2}-1$ if and only if the corresponding parameters p, q, r and s satisfy one of the conditions 1.–10. from Lemma 7 or one of the conditions 1.–5. from Lemma 8.

This completes the proof.

REFERENCES

1. D. Cao, H. Yuan: *Graphs characterized by the second eigenvalue.* J. Graph Theory, **17** (1993), No. 3, 325–331.
2. D. M. Cvetković: *Graphs and their spectra* (Thesis). Univ. Beograd Publ. Elektrotehn. Fak., Ser. Mat. Fiz., No. **354**–No. **356** (1971), 1–50.
3. D. M. Cvetković, M. Doob, H. Sachs: *Spectra of graphs–theory and application.* Academic Press, New York, 1980.
4. M. Petrović: *On graphs with exactly one eigenvalue less than* –1. J.Comb.Theory (B), **52** (1991), 102–112.
5. J. H. Smith: *Some properties of the spectrum of a graph.* In: Combinatorial Structures and Their Applications, Gordon and Breach, Science Publ., Inc., New York-London-Paris 1970, 403–406.
6. E. S. Wolk: *A note on the comparability graph of a tree.* Proc. of A.M.S. **16** (1965), 17–20.

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