UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. 4 (1993), 35–42.

ON SIGN OF THE MATHEMATICAL EXPECTATION OF AN ERROR IN SIGNAL TRANSMISSION

Ilija Lazarević

The error probability of signal transmission in Telecommunication systems can not be expressed in terms of elementary functions. The mathematical formulas are usually very complex ones. In recent years, various upper and lower bound and computer algorithms have been developed to compute the error probability. In this paper we investigate the sign of the integral (4), which appears as a kernel in the formula for mathematical expectation of the error in a binary telecommunication system. We are considering a real domain only, because we are interested in certain inequalities. Knowing the sign of this kernel, some estimates for the error probability can be found.

0. INTRODUCTION

A basic theoretical problem in Telecommunications is to find an error probability in digital signal transmission. Even in the most simple cases, this problem is not easy to handle. Instead of using exact values, the usual procedure is to find upper and lower bounds.

V. PRABHY [**5**] suggested a modified CHERNOFF error for determining the error probability of a pulse modulated signal with a presence of a Gaussian noise and the interference. It was shown in [**5**] that the error probability in the case of a binary system can be expressed as follows:

$$P = \exp(-\lambda\beta + \beta^2\sigma^2/2)I, \tag{1}$$

[0]1991 Mathematics Subject Classification: 33B20

where

$$I = \frac{1}{2\pi}\langle \exp(\eta\beta)K\rangle \tag{2}$$

$$K = \int_{-\infty}^{+\infty} \frac{\beta\cos(x(\eta-(\lambda-\beta\sigma^2))) + x\sin(x(\eta-(\lambda-\beta\sigma^2)))}{\beta^2+x^2}\exp(-x^2\sigma^2/2)\,\mathrm{d}x, \tag{3}$$

where input symbols, chosen with equal a priori probability from a binary aphabet $\{\lambda, -\lambda\}$, are corrupted by additive zero-mean Gaussian noise of variance σ^2 and independent additive interference $\eta(t)$, such as intersymbol and cochanel interefer-ence. Notation $\langle\cdot\rangle$ is used for the mathematical expectation, and the parameter β is a positive real number.

Upper and lower bounds on error probability are given in [**5**] in terms of the moment generating functions of η. The bounds obtained here are not the best possible ones. A better lower bound is conjectured in [**6**]. Further, in this paper the authors noted that, under the assumptions from [**5**], when the PRABHU bound is minimal, then the sine part of the integral (3) is positive. However, this is confirmed on some particular cases using computer evaluations. The aim of this paper is further investigation of the sign of the sine part of (3).

1. SIGN OF THE STATISTICAL KERNEL

We will investigate the kernel

$$\int_{-\infty}^{+\infty} \frac{x\sin(\eta-\mu)x}{x^2+\beta^2}e^{-\sigma^2x^2/2}\,\mathrm{d}x, \tag{4}$$

which appears in (3), with $\mu = \lambda - \beta\sigma^2$.

We will show that it is possible to determine the sign of (4) at least in some particular cases. Let

$$\lambda = \eta - \mu\ , \quad a = \frac{1}{2}\sigma^2, \tag{5}$$

and let us investigate the function $J : \mathbf{R} \mapsto \mathbf{R}$, defined by

$$J(\lambda) = \int_{-\infty}^{+\infty} \frac{x\sin\lambda x}{x^2+\beta^2}\,e^{-ax^2}\,\mathrm{d}x, \tag{6}$$

for $\lambda > 0, a > 0, \beta > 0$. Integral in (6) converges uniformly for $a > 0$. It is easy to see that

$$J'(\lambda) = \int_{-\infty}^{+\infty} \frac{x^2\cos\lambda x}{x^2+\beta^2}e^{-ax^2}\,\mathrm{d}x \tag{7}$$

and

$$J''(\lambda) - \beta^2 J(\lambda) = b(\lambda), \tag{8}$$

where

$$b(\lambda) = -\int_{-\infty}^{+\infty} e^{-ax^2} x \sin \lambda x \, \mathrm{d}x. \tag{9}$$

Starting from a well known result (see, for instance, [**4**])

$$\int_{-\infty}^{+\infty} e^{-x^2} \cos \lambda x \, \mathrm{d}x = \sqrt{\pi} e^{-\frac{\lambda^2}{4}}, \tag{10}$$

letting $x = t/\sqrt{a}$ and differentiating with respect to λ, we have:

$$b(\lambda) = -\frac{\sqrt{\pi}\lambda}{2a\sqrt{a}} e^{-\frac{\lambda^2}{4a}}. \tag{11}$$

Therefore, the function $\lambda \mapsto J(\lambda)$ satisfies the differential equation (8), with $b(\lambda)$ defined by (8). Besides, we have

$$J(0) = 0 \tag{12}$$

and

$$J'(0) = \int_{-\infty}^{+\infty} \frac{x^2}{x^2 + \beta^2} e^{-ax^2} \, \mathrm{d}x = 2 \left(\int_{0}^{+\infty} e^{-ax^2} \, \mathrm{d}x - \beta^2 I \right), \tag{13}$$

where

$$I = \int_{0}^{+\infty} \frac{e^{-ax^2}}{x^2 + \beta^2} \, \mathrm{d}x. \tag{14}$$

To find I, note that $\frac{1}{x^2+\beta^2} = \int_0^{+\infty} e^{-(x^2+\beta^2)t} \, \mathrm{d}t$ and, therefore

$$I = \int_{0}^{+\infty} \left(\int_{0}^{+\infty} e^{-(x^2+\beta^2)t} \, \mathrm{d}t \right) e^{-ax^2} \, \mathrm{d}x = \int_{0}^{+\infty} \left(\int_{0}^{+\infty} e^{-(a+t)x^2} \, \mathrm{d}x \right) e^{-\beta^2 t} \, \mathrm{d}t.$$

By a change of variable $t = y^2/\beta^2 - a$, we get

$$I = \frac{\sqrt{\pi}}{2\sqrt{a+t}} \int_{0}^{+\infty} e^{-\beta^2 t} \, \mathrm{d}t = \frac{\sqrt{\pi} e^{a\beta^2}}{\beta} \int_{\beta\sqrt{a}}^{+\infty} e^{-y^2} \, \mathrm{d}y = \frac{\pi e^{a\beta^2}}{2\beta} (1 - \mathrm{erf}(\beta\sqrt{a})),$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x e^{-t^2}\,\mathrm{d}t.$$

If we solve the differential equation (8) with conditions (12) and (13), we get (using variations of constants):

$$J(\lambda) = \frac{1}{2\beta} e^{\beta\lambda} D_1(\lambda)\xi\lambda - \frac{1}{2\beta} e^{-\beta\lambda} D_2(\lambda)\xi\lambda + \frac{\xi}{\beta}\mathrm{sh}\beta\lambda, \tag{16}$$

where

$$D_1(\lambda) = -\frac{\sqrt{\pi}}{2a\sqrt{a}} \int\limits_0^\lambda \lambda e^{-\frac{\lambda^2}{4a}-\beta\lambda}\,\mathrm{d}\lambda, \tag{17}$$

$$D_2(\lambda) = -\frac{\sqrt{\pi}}{2a\sqrt{a}} \int\limits_0^\lambda \lambda e^{-\frac{\lambda^2}{4a}+\beta\lambda}\,\mathrm{d}\lambda, \tag{18}$$

$$\xi = \sqrt{\frac{\pi}{a}} - \pi\beta e^{a\beta^2}(1 - \operatorname{erf}(\beta\sqrt{a})). \tag{19}$$

Functions D_1 and D_2 may be expressed in terms of the error function, by an appropriate change of variables in integrals appearing in (17) and in (18):

$$D_1(\lambda) = \frac{\sqrt{\pi}}{\sqrt{a}}(e^{-\frac{\lambda^2}{4a}-\beta\lambda} - 1) + \pi\beta e^{a\beta^2}\left(\operatorname{erf}\left(\frac{\lambda}{2\sqrt{a}} + \beta\sqrt{a}\right) - \operatorname{erf}(\beta\sqrt{a})\right). \tag{20}$$

$$D_2(\lambda) = \frac{\sqrt{\pi}}{\sqrt{a}}(e^{-\frac{\lambda^2}{4a}+\beta\lambda} - 1) - \pi\beta e^{a\beta^2}\left(\operatorname{erf}\left(\frac{\lambda}{2\sqrt{a}} - \beta\sqrt{a}\right) + \operatorname{erf}(\beta\sqrt{a})\right). \tag{21}$$

Finally, using (16), (19), (20) and (21), we obtain the following expression for J:

$$J(\lambda) = \pi e^{a\beta^2}\left(\frac{1}{2}\left(e^{\beta\lambda}\operatorname{erf}\left(\frac{\lambda}{2\sqrt{a}} + \beta\sqrt{a}\right) + e^{-\beta\lambda}\operatorname{erf}\left(\frac{\lambda}{2\sqrt{a}} - \beta\sqrt{a}\right)\right) - \mathrm{sh}\beta\lambda\right). \tag{22}$$

Formula (22) can be also obtained by methods of Complex Analysis applied to the integrals $J(\lambda)$ and $J'(\lambda)$ (see, for example, [**8**], p. 511).

It seems that the expressions of the form $A(t)\operatorname{erf}(\alpha t + \beta) + B(t)\operatorname{erf}(\alpha t - \beta)$, like the one in (22), naturally appear in some applied problems (see [**7**]).

Now we can formulate the following theorem.

Theorem 1. *For $a > 0$ the following holds*:

$$\operatorname{sgn} J(\lambda) = \operatorname{sgn}\lambda \tag{23}$$

$$J(0) = 0\ ; \quad J'(0) > 0\ ; \quad J(+\infty) = 0_+. \tag{24}$$

Proof. The form of (22) is convenient for investigation of the sign of J. Since J is an odd function, it suffices to assume $\lambda \geq 0$. Letting $\beta\lambda = t$ and $1/(2\beta\sqrt{a}) = \alpha$, the inequality $J(\lambda) \geq 0$ becomes equivalent to

$$\frac{1}{2}\left(e^t \operatorname{erf}\left(\alpha t + \frac{1}{2\alpha}\right) + e^{-t} \operatorname{erf}\left(\alpha t - \frac{1}{2\alpha}\right)\right) \geq \operatorname{sh} t. \tag{25}$$

In [1] it has been shown that (25) holds for every real t and α of the same sign. Therefore, we have (23). Relations in (24) are straightforward consequences of (6) and (13). The third relation can be obtained by taking limits in (22) and by using the known value of the Poisson integral. □

Corolary 1. By a mean value theorem we have

$$J(\lambda) = 2\int_0^{+\infty} \frac{x \sin \lambda x}{x^2+\beta^2} e^{-ax^2}\,\mathrm{d}x = 2M(\lambda)\int_0^{+\infty} e^{-ax^2}\,\mathrm{d}x = 2M(\lambda)\sqrt{\frac{\pi}{a}}\,, \tag{26}$$

where $M(\lambda)$ is the mean value in $[0, +\infty)$ of the function $g(x) = \frac{x \sin \lambda x}{x^2+\beta^2}$. □

Corolary 2. For $\lambda > 0$ we proved that $J(\lambda > 0$; therefore $M(\lambda) > 0$. Further, it is easy to see that $|g(\lambda)| \leq 1/(2\beta)$. So, we have

$$0 \leq J(\lambda) \leq \frac{1}{2\beta}\sqrt{\frac{\pi}{a}} \quad \text{for } \lambda \geq 0\ . \tag{27}$$

It is clear from (7) that $J'(0) > 0$. However, if we find the derivative of J according to (22), we get

$$J'(0) = \sqrt{\frac{\pi}{a}} - \pi\beta e^{a\beta^2}\left(1 - \operatorname{erf}(\beta\sqrt{a})\right)\,, \tag{28}$$

and it is not at all obvious what sign of J should be. To see that from (28) we can use an inequality due to R. D. Gordon [**2**]: $R(x) \leq 1/x$, where $R(x)$ is so called Mill's ratio :

$$R(x) = e^{x^2/2}\int_x^{+\infty} e^{-t^2/2}\,\mathrm{d}x.$$

Indeed, using usual transformations, we have $1 - \operatorname{erf}(y) = \sqrt{\frac{\pi}{a}}e^{-y^2}R(y\sqrt{2})$, and, by (28) we have

$$J'(0) = \sqrt{\frac{\pi}{a}} - \beta\sqrt{2\pi}R(\beta\sqrt{2a}) > \sqrt{\frac{\pi}{a}} - \beta\sqrt{2\pi}\frac{1}{\beta\sqrt{2a}} = 0.$$

Conversely, using (7) and (28), we can derive the result of Gordon. □

2. SIGN OF MATHEMATICAL EXPECTATION

In this section we will use the statistical kernel J to investigate the sign of the expectation.

Mathematical expectation from (2) can be written in the form

$$E = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{\beta\eta} w(\eta) J(\eta - \mu) \, \mathrm{d}\eta \,. \tag{29}$$

We will assume that w is non-negative integrable even function on $\mathbf{R}$ and $\beta > 0$. Letting $\eta = t + \mu$ in (29) we get

$$E = \frac{e^{\beta\mu}}{2\pi} \left(\int_{-\infty}^{0} e^{\beta t} w(t + \mu) J(t) \, \mathrm{d}t + \int_{0}^{+\infty} e^{-\beta t} w(t + \mu) J(t) \, \mathrm{d}t \right)$$

and further, since J is an odd function :

$$E = \frac{e^{\beta\mu}}{2\pi} \int_{0}^{+\infty} (e^{\beta t} w(t + \mu) - e^{-\beta t} w(t - \mu)) J(t) \, \mathrm{d}t. \tag{30}$$

This form is useful for determining necessary (and sometimes also sufficient) conditions for non-negativity of E. One such condition is

$$e^{\beta t} w(t + \mu) - e^{-\beta t} w(t - \mu) \geq 0 \qquad (t \geq 0). \tag{31}$$

If w does not have real zeros, this condition takes the form

$$\frac{w(t - \mu)}{w(t + \mu)} \leq e^{2\beta t} \qquad (t \geq 0). \tag{32}$$

These conditions can be used as tests in certain particular cases.

Example 1. Let w be an even non-decreasing function on $\mathbf{R}^+$ and let $\mu > 0$. Then by the obvious inequality $|t - \mu| \leq t + \mu$ we have $w(t - \mu) = w(|t - \mu|) \leq w(t + \mu)$, and the condition (32) is satisfied.

Example 2. Let $w(\eta) = 1 + \sin^2 \eta$, $\mu = \pi/2$. In this case we have $w(t - \mu) = w(t + \mu)$, and the condition (32) is satisfied.

Example 3. Let $w(\eta) = 1/(\eta^2 + 1), \mu > 0$. Then we have:

$$\frac{w(t - \mu)}{w(t + \mu)} = \frac{(t + \mu)^2 + 1}{(t - \mu)^2 + 1} = 1 + \frac{4\mu t}{(t - \mu)^2 + 1} \,.$$

Using the inequalities $e^{2\beta t} \geq 1 + 2\beta t + 2\beta^2 t^2$, $\frac{1}{(t-\mu)^2+1} \leq 1$, we have $\Delta = e^{2\beta t} - \frac{w(t-\mu)}{w(t+\mu)} \geq 2t(\beta - 2\mu)$, so for $\beta \geq 2\mu$, the inequality (31) holds.

Example 4. Let $w(\eta) = g(\eta)$ if $|\eta| \leq c$ and $w(\eta) = 0$ elsewhere; assume that g is an even non-negative function. We can separate 3 cases (see Fig.1–Fig.3):

1° Case $\mu \geq c$. In this case the formula (30) becomes

$$E = \frac{e^{\beta\mu}}{2\pi} \int_{\mu-c}^{\mu+c} \Delta(t) J(t)\, \mathrm{d}t \quad \Delta(t) = e^{\beta t} w(t+\mu) - e^{-\beta t} w(t-\mu).$$

In this case $\Delta(t) \leq 0$ for $t \in [\mu - c, \mu + c]$ and therefore $E < 0$.

2° Case $\mu \leq -c$. In this case we integrate over the interval $[-\mu - c, -\mu + c]$, and $\Delta(t) \geq 0$ there, so $E > 0$.

3° Case $-c < \mu < c$. If we assume $\mu > 0$, then we integrate over $[0, \mu + c]$. In this case $\Delta(t)$ is negative on $[-\mu + c, \mu + c]$, but need not be of a constant sign on $[0, -\mu + c]$. So, in this case we can not reach any conclusion regarding the sign of E.

Fig. 1

Fig. 2

Fig. 3

Acknowledgement. The author thanks Prof. G. Petrović for useful discussions on Information theory.

REFERENCES

1. I. Lazarević, M. Merkle: *An inequality involving erf and hyperbolic functions.* Bull. Mathematique (Romania), No. 3–4 (1992).
2. R. D. Gordon: *Values of Mills' ratio of area to bounding ordinate and of the normal probability integral for large values of the argument.* Ann. Math. Statistics, **12** (1941), 364–366.
3. D.S. Mitrinović (with P. M. Vasić): *Analytic Inequalities.* 1970.
4. G. M. Fihtengolc: *Kurs differencialnogo i integralnogo isčisleniya.* Moskva, 1969.
5. V. Prabhu: *Modified Chernoff bounds for PAM systems with noise and interference.* IEEE Trans. Inform. Theory, Vol. IT-**28**, No. 1 (1982), 95–99.
6. Z. Nikolić, G. Petrović, P. Kostić. *O Prabhuovim granicama za procenu greške po BITU.* 22. Konferencija ETANa, Zagreb (1990).
7. Problem $91 - 1^*$ and $91 - 4^*$. Siam Review, Vol. **33**, No.1 (1991).
8. I. S. Gradstein, I. M. Ryžik: *Tables of integrals, sums, series and products* (Russian), Nauka, Moskva, 1971.

Faculty of Electrical Engineering,
University of Belgrade,
P.O.Box 816, 11001 Belgrade,
Yugoslavia

(Received November 5, 1992)
(Revised October 18, 1993)