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TEN NEW ORDERS FOR HADAMARD MATRICES OF SKEW TYPE

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We prove that there exist skew type Hadamard matrices of order $4n$ for $n = 39$, 49, 65, 93, 121, 129, 133, 217, 219 and 267 which have not been constructed so far. We also obtain a Hadamard matrix of order 532 of maximal excess 12236. We mention that very recently we have constructed skew type Hadamard matrices of order $4n$ for $n = 37$, 43, 67, 113, 127, 157, 163, 181 and 241, see [1] and [2].

1. INTRODUCTION

1. A *Hadamard matrix* of order m is a $(1, -1)$-matrix H of order m satisfying $HH^{\mathrm{T}} = mI_m$. (X^{T} denotes the transpose of a matrix X, and I_m the identity matrix of order m.) The order m of a HADAMARD matrix H must be 1, 2 or a multiple of 4, $m = 4n$. A $(1, -1)$-matrix A of oder m is said to be of *skew type* if $A + A^{\mathrm{T}} = 2I_m$. A skew *Hadamard matrix* is a HADAMARD matrix of skew type.

It has been conjectured that HADAMARD matrices as well as skew HADAMARD matrices exist for all orders m which are multiples of 4. According to [**2**] and [**7**], there are 59 odd integers $n < 300$ for which the existence of skew HADAMARD matrices of order $m = 4n$ is not yet established. These numbers are the following:

$$(1)\quad \begin{gathered} 39, 47, 49, 59, 65, 69, 81, 89, 93, 97, 101, 103, 107, 109, 119, 121, 129, 133, 145, \\ 149, 151, 153, 167, 169, 177, 179, 191, 193, 201, 205, 209, 213, 217, 219, 223, \\ 225, 229, 233, 235, 239, 245, 247, 249, 251, 253, 257, 259, 261, 265, 267, 269, \\ 275, 277, 283, 285, 287, 289, 295, 299. \end{gathered}$$

In this paper we prove the existence of skew HADAMARD matrices of order $m = 4n$ for $n = 39$, 49, 65, 93, 121, 129, 133, 217, 219 and 267. Hence these ten numbers should be removed from the list (1), and consequently the revised list will contain

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only 49 numbers. Note that all these ten numbers are composite, while the n's dealt with in [**1**] and [**2**] were all prime. For information about the known infinite series and previous work on the orders of skew Hadamard matrices, we refer the reader to the book [**3**] and the papers [**6**] and [**8**].

2. We shall identify the integer $i \in \{0, 1, 2, \ldots, n-1\}$ with the corresponding residue class modulo n. We say that subsets $S_1, \ldots, S_k$ of $\{0, 1, 2, \ldots, n-1\}$ are

$$k - (n; n_1, \ldots, n_k; \lambda)$$

supplementary difference sets modulo n (abbreviated as *sds*) if $|S_i| = n_i$ for all i and for each nonzero residue r modulo n we have $\lambda_1(r) + \ldots + \lambda_4(r) = \lambda$, where $\lambda_i(r)$ is the number of solutions of the congruence

$$x - y \equiv r \pmod{n}$$

with $x, y \in S_i$. Our construction of skew Hadamard matrices is based on some new $4 - (n; n_1, \ldots, n_4; \lambda)$ sds's with $\lambda = n_1 + \cdots + n_4 - n$. They were found with a help of a computer.

We say that a subset S of $\{1, 2, \ldots, n-1\}$ is of skew type if

$$i \in S \iff -i \notin S.$$

They exist if and only if n is odd. Given any subset S of $\{1, 2, \ldots, n-1\}$ we denote by $A_S = (a_{ij})$ the circulant $(1, -1)$-matrix of order n, $0 \le i, j \le n-1$, whose first row is given by

$$a_{0,j} = \begin{cases} -1 & \text{if } j \in S, \\ 1 & \text{if } j \notin S. \end{cases}$$

Since $0 \notin S$, all diagonal entries of A_S are 1's. A_S is of skew type if and only if S is of skew type. For later use we introduce the permutation matrix $R = (r_{ij})$ of order n, $0 \le i, j \le n-1$, such that

$$r_{ij} = \begin{cases} 1 & \text{if } i + j \equiv -1 \pmod{n}, \\ 0 & \text{otherwise} \end{cases}$$

3. We say that a $(1, -1)$-matrix H of order $m = 4n$ is of Goethals-Seidel type if

$$H = \begin{pmatrix} A_1 & A_2R & A_3R & A_4R \\ -A_2R & A_1 & -A_4^{\mathrm{T}}R & A_3^{\mathrm{T}}R \\ -A_3R & A_4^{\mathrm{T}}R & A_1 & -A_2^{\mathrm{T}}R \\ -A_4R & -A_3^{\mathrm{T}}R & A_2^{\mathrm{T}}R & A_1 \end{pmatrix} \tag{2}$$

where A_i are circulant matrices of order n and R is the matrix defined in the previous section. Such H is a Hadamard matrix if and only if

$$\sum_{i=1}^{4} A_i A_i^{\mathrm{T}} = 4nI_n,$$

see [**5**]. Furthermore H is of skew type if and only if A_1 is of skew type. All HADAMARD matrices constructed in this paper are of Goethals-Seidel type.

Let $S = \{S_1, S_2, S_3, S_4\}$ be a 4-tuple of subsets of $\{1, 2, \ldots, n-1\}$ and let H_S be the matrix H above with $A_i = A_{S_i}$. We let $n_i = |S_i|$. The following result is well known.

Theorem 1 (See [**5**]). *The matrix H_S is a* HADAMARD *matrix if and only if S_1, S_2, S_3, S_4 are $4 - \left(n; n_1, n_2, n_3, n_4, \sum_{i=1}^{4} n_i - n\right)$ supplementary difference sets modulo n. Furthermore H_S is of skew type if and only if S_1 is of skew type.*

The number $r_i = n - 2n_i$ is the row sum of the circulant A_i. If n is odd, each r_i is an odd integer. We shall refer to the quadruple (r_i), arranged in decreasing order, as the *type* of the sds S. If H_S is a HADAMARD matrix then

$$\sum_{i=1}^{4} r_i^2 = 4n. \tag{3}$$

4. Let n be one of the ten integers mentioned in Section 1. We denote by G the multiplicative group of all residue classes modulo n which are relatively prime to n. We choose a subgroup H of order k for some odd integer $k > 1$. G and H act (by multiplication modulo n) on the set a of nonzero residue classes modulo n which we identify with the set of integers $\{1, 2, \ldots, n-1\}$. We enumerate the orbits of H in A as $\alpha_0, \alpha_1, \ldots, \alpha_{b-1}$. In all cases that we consider b is even and $-1 \cdot \alpha_i \neq \alpha_i$ for all i. Our enumeration is such that

$$\alpha_{2i+1} = -1 \cdot \alpha_{2i}, \qquad 0 \leq i < b/2,$$

and so it suffices to list the orbits α_{2i} only. The sets S_i are constructed as unions of orbits α_i. By Theorem 1, in order to construct a skew HADAMARD matrix of order $m = 4n$ if suffices to produce four sets S_i with S_1 of skew type which are $4 - \left(n; n_1, \ldots, n_4; \sum n_i - n\right)$ sds.

Two $k - (n; n_1, \ldots, n_k; \lambda)$ sds's, say (S_i) and (S_i'), are said to be *equivalent* if there exist integers s, t, with $t \in G$, and a permutation σ such that $s + tS_i = S'_{\sigma(i)}$ for all i.

Theorem 2. *There exist skew* HADAMARD *matrices of order $4n$ for $n = 39$, 49, 65, 93, 121, 129, 133, 217, 219 and 267.*

We have to exhibit the required sds's. For each n mentioned in the theorem, we shall list several nonequivalent sds's and their types. (We did not try to find all of them.) For at least one of the listed sds's, the first set, S_1, is of skew type. As each S_i has the form

$$S_i = \bigcup_{j \in J_i} \alpha_j,$$

it suffices to enumerate the orbits and list the index sets J_i. We remark that in the case $n = 65$ we have found sds's corresponding to all five possible decompositions (3) of $4n$ as a sum of four odd squares. In the case $n = 133$ the first two sds's listed give rise to a HADAMARD matrix of order 532 of maximal excess.

Case $n = 39$. Let $H = \{1, 16, 22\}$ be the subgroup of G of order 3. The orbits α_{2i} are:

$$\alpha_0 = H,\ \alpha_2 = 2H,\ \alpha_4 = 3H,\ \alpha_6 = 4H,\ \alpha_8 = 6H,\ \alpha_{10} = 8H,\ \alpha_{12} = \{13\}.$$

Recall that $\alpha_{2i+1} = -1 \cdot \alpha_{2i}$. In this case we list three sds's.

(a) $4-(39; 19, 14, 17, 18; 29)$ sds of type $(11, 5, 3, 1)$, with S_1 of skew type.

$J_1 = \{1, 3, 5, 6, 8, 10, 12\}$, $J_2 = \{0, 1, 5, 8, 12, 13\}$,
$J_3 = \{1, 3, 4, 7, 9, 12, 13\}$, $J_4 = \{0, 1, 2, 3, 7, 8\}$.

(b) $4-(39; 19, 15, 16, 17; 28)$ sds of type $(9, 7, 5, 1)$, with S_1 of skew type.

$J_1 = \{0, 2, 4, 7, 9, 11, 12\}$, $J_2 = \{1, 6, 7, 9, 11\}$,
$J_3 = \{1, 2, 3, 6, 9, 13\}$, $J_4 = \{3, 5, 6, 7, 10, 12, 13\}$.

(c) $4-(39; 16, 16, 16, 18; 27)$ sds of type $(7, 7, 7, 3)$.

$J_1 = \{0, 2, 4, 8, 10, 13\}$, $J_2 = \{1, 2, 3, 8, 9, 13\}$,
$J_3 = \{1, 3, 5, 10, 11, 12\}$, $J_4 = \{0, 7, 8, 9, 10, 11\}$.

Case $n = 49$. $H = \{1, 18, 30\}$ is the subgroup of G of order 3. The orbits α_{2i} are:

$$\alpha_0 = H,\quad \alpha_2 = 2H,\quad \alpha_4 = 3H,\quad \alpha_6 = 4H,\quad \alpha_8 = 6H,$$
$$\alpha_{10} = 7H,\quad \alpha_{12} = 9H,\quad \alpha_{14} = 12H.$$

The quadruples (a), (b), (c), and (d), given below, define nonequivalent $4-(49; 24, 18, 24, 27; 44)$ sds's of type $(13, 1, 1, -5)$, with S_1 of skew type.

(a) $J_1 = \{1, 2, 5, 7, 8, 10, 13, 14\}$, $J_2 = \{4, 5, 6, 7, 10, 11\}$,
$J_3 = \{0, 1, 2, 4, 6, 7, 12, 14\}$, $J_4 = \{1, 2, 3, 5, 6, 10, 12, 13, 14\}$.

(b) $J_1 = \{0, 3, 4, 7, 8, 10, 12, 14\}$, $J_2 = \{4, 5, 6, 7, 10, 15\}$,
$J_3 = \{0, 2, 7, 8, 9, 10, 14, 15\}$, $J_4 = \{0, 2, 5, 6, 11, 12, 13, 14, 15\}$.

(c) J_1 and J_2 are the same as in (b) and
$J_3 = \{0, 2, 4, 10, 11, 13, 14, 15\}$, $J_4 = \{0, 2, 3, 6, 7, 8, 13, 14, 15\}$.

(d) J_1 and J_2 as in (b),
$J_3 = \{1, 3, 5, 10, 11, 12, 14, 15\}$, $J_4 = \{1, 3, 4, 7, 8, 9, 10, 11, 15\}$.

The next four quadruples define nonequivalent $4-(49; 24, 21, 27, 30; 53)$ sds's of type $(7, 1, -5, -11)$, with S_1 of skew type.

(e) $J_1 = \{1, 2, 5, 7, 8, 10, 13, 14\}$, $J_2 = \{0, 2, 4, 5, 7, 8, 9\}$,
$J_3 = \{0, 2, 4, 6, 7, 10, 11, 13, 15\}$, $J_4 = \{0, 1, 2, 3, 5, 8, 10, 11, 12, 15\}$.

(f) J_1 as in (e), $J_2 = \{0, 4, 5, 7, 8, 9, 13\}$,
$J_3 = \{0, 2, 7, 8, 9, 10, 12, 14, 15\}$, $J_4 = \{0, 1, 4, 6, 7, 8, 10, 11, 12, 13\}$.

(g) J_1 and J_4 as in (e), $J_2 = \{1, 3, 4, 5, 6, 8, 9\}$, $J_3 = \{1, 3, 6, 8, 9, 11, 13, 14, 15\}$.

(h) J_1, J_2, J_4 as in (e), $J_3 = \{1, 3, 5, 6, 7, 10, 11, 12, 14\}$.

The next two quadruples define nonequivalent $4-(49;21,21,21,21;35)$ sds's of type $(7,7,7,7)$.

(i) $J_1=\{2,3,4,5,12,13,15\}$, $J_2=\{0,1,2,3,6,7,15\}$,
$J_3=\{0,5,6,11,13,14,15\}$, $J_4=\{4,5,6,7,10,11,15\}$.

(j) J_3 and J_4 as in (i), $J_1=\{0,1,6,7,8,9,15\}$, $J_2=\{0,1,2,3,6,7,14\}$.

The next two quadruples define nonequivalent $4-(49;27,27,27,30;62)$ sds's of type $(-5,-5,-5,-11)$.

(k) $J_1=\{0,4,5,6,7,8,9,12,14\}$, $J_2=\{0,1,3,4,6,9,11,12,13\}$,
$J_3=\{1,3,4,5,6,7,8,10,11\}$, $J_4=\{0,1,5,6,7,9,10,13,14,15\}$.

(l) J_2, J_3, J_4 as in (k), $J_1=\{1,4,5,6,7,8,9,13,15\}$.

Case $n=65$. $H=\{1,16,61\}$ is the subgroup of G of order 3. The orbits α_{2i} are:

$$\alpha_0=H,\quad \alpha_2=2H,\quad \alpha_4=3H,\quad \alpha_6=5H,\quad \alpha_8=6H,\quad \alpha_{10}=7H,$$
$$\alpha_{12}=9H,\quad \alpha_{14}=10H,\quad \alpha_{16}=11H,\quad \alpha_{18}=\{13\},\quad \alpha_{20}=22H,\quad \alpha_{22}=\{26\}.$$

The first wo quadruples, (a) and (b), define $4-(65;32,25,34,35;61)$ sds's of type $(15,1,-3,-5)$, with S_1 of skew type.

(a) $J_1=\{1,3,5,6,8,10,13,14,17,18,20,22\}$,
$J_2=\{0,3,7,10,16,17,18,20,21\}$,
$J_3=\{2,4,6,8,9,10,14,15,16,17,18,20\}$,
$J_4=\{5,7,8,9,11,12,13,14,16,18,19,20,21\}$.

(b) J_1, J_2, J_3 as in (a), $J_4=\{4,6,8,9,10,12,13,15,17,18,19,20,21\}$.

The next two quadruples define nonequivalent $4-(65;32,26;28,34;55)$ sds's of type $(13,9,1,-3)$, with S_1 of skew type.

(c) $J_1=\{0,2,4,6,9,10,12,15,17,19,21,22\}$,
$J_2=\{1,3,6,9,10,13,20,21,22,23\}$,
$J_3=\{0,1,4,7,13,14,17,20,21,23\}$,
$J_4=\{2,3,5,6,8,10,11,12,14,15,16,23\}$.

(d) J_1 and J_4 as in (c), $J_2=\{0,2,7,8,11,12,20,21,22,23\}$,
$J_3=\{4,5,8,10,12,13,14,15,21,22\}$.

(e) $4-(65;31,31,38,38;73)$ sds of type $(3,3,-11,-11)$.

$J_1=\{0,1,3,5,7,8,9,10,12,15,18\}$,
$J_2=\{0,2,6,10,11,12,15,16,20,21,23\}$,
$J_3=\{1,2,4,5,7,12,13,15,16,17,20,21,22,23\}$,
$J_4=\{2,3,4,5,7,8,9,11,12,14,16,17,18,19\}$.

(f) $4-(65;31,34,38,38;76)$ sds of type $(3,-3,-11,-11)$. J_3 and J_4 as in (e),

$$J_1 = \{0,4,5,6,8,10,12,14,17,20,22\},$$
$$J_2 = \{0,4,5,6,7,8,12,14,17,20,21,23\}.$$

(g) $4-(65;34,34,38,38;79)$ sds of type $(-3,-3,-11,-11)$. J_3 and J_4 as in (e),

$$J_1 = \{0,3,5,7,9,12,13,14,17,20,21,23\},$$
$$J_2 = \{1,2,3,4,7,8,10,12,15,16,17,23\}.$$

(h) $4-(65;28,29,31,38;61)$ sds of type $(9,7,3,-11)$.

$$J_1 = \{1,4,5,6,9,13,14,16,20,22\},$$
$$J_2 = \{1,2,3,4,7,9,11,13,16,22,23\},$$
$$J_3 = \{0,1,2,4,14,15,16,17,20,21,23\},$$
$$J_4 = \{2,4,7,8,9,11,12,13,15,16,17,18,19,20,21\}.$$

(i) $4-(65,29,31,37,38;70)$ sds of type $(7,3,-9,-11)$.

$$J_1 = \{1,2,3,4,8,9,10,15,16,18,19\},$$
$$J_2 = \{2,4,6,11,13,14,16,17,20,21,23\},$$
$$J_3 = \{0,1,4,5,7,10,11,12,13,15,17,18,21\},$$
$$J_4 = \{0,1,2,3,4,7,8,10,11,14,16,17,22,23\}.$$

(j) $4-(65;29,34,37,38;73)$ sds of type $(7,-3,-9,-11)$.

$$J_1 = \{0,4,6,10,12,13,17,20,21,22,23\},$$
$$J_2 = \{0,3,5,7,9,12,13,14,17,20,21,23\},$$
$$J_3 = \{3,4,5,7,8,9,11,12,14,16,17,18,20\},$$
$$J_4 = \{0,1,2,3,4,6,8,12,13,14,18,19,20,21\}.$$

(k) $4-(65:28,28,29,29;49)$ sds of type $(9,9,7,7)$.

$$J_1 = \{0,2,3,7,9,14,16,20,21,23\},$$
$$J_2 = \{1,2,6,10,13,14,16,20,21,23\},$$
$$J_3 = \{0,1,4,10,11,12,13,15,17,18,19\},$$
$$J_4 = \{2,3,5,6,8,15,16,17,21,22,23\}.$$

(l) $4-(65;28,28,36,36;63)$ sds of type $(9,9,-7,-7)$.

$$J_1 = \{0,4,5,6,9,12,16,20,21,23\},$$
$$J_2 = \{1,4,6,7,8,9,10,13,16,18\},$$
$$J_3 = \{0,1,3,6,7,10,13,15,16,17,20,21\},$$
$$J_4 = \{1,2,3,7,11,13,14,15,16,17,20,21\}.$$

(m) $4-(65;29,29,37,37;67)$ sds of type $(7,7,-9,-9)$. J_3 and J_4 as in (k),

$$J_1 = \{0,1,2,3,4,5,7,8,9,10,14,19,21\},$$
$$J_2 = \{1,2,3,4,6,8,13,14,16,17,20,21,23\}.$$

Note that we have found an sds for each of the 5 different decompositions of $4 \cdot 65 = 260$ as a sum of four odd squares.

Case $n = 93$. $H = \{1,4,16,64,70\}$ is the subgroup of G of order 5. The orbits α_{2i} are:

$$\alpha_0 = H, \quad \alpha_2 = 2H, \quad \alpha_4 = 3H, \quad \alpha_6 = 5H, \quad \alpha_8 = 7H, \quad \alpha_{10} = 9H,$$
$$\alpha_{12} = 10H, \quad \alpha_{14} = 14H, \quad \alpha_{16} = 15H, \quad \alpha_{18} = \{31\}.$$

(a) $4-(93;46,37,45,46;81)$ sds of type $(19,3,1,1)$, with S_1 of skew type.

$$J_1 = \{0,3,4,6,9,10,12,14,17,18\},$$
$$J_2 = \{2,3,4,5,9,13,15,18,19\},$$
$$J_3 = \{1,2,3,4,5,6,7,8,16\},$$
$$J_4 = \{1,4,6,11,12,13,15,16,17,18\}.$$

(b) $4-(93;46,37,45,47;82)$ sds of type $(19,3,1,-1)$, with S_1 of skew type. J_1, J_2, J_3 as in (a), $J_4 = \{1,4,7,9,10,14,15,16,17,18,19\}$.

(c) $4-(93;46,45,46,56;100)$ sds of type $(3,1,1,-19)$, with S_1 of skew type.

$$J_1 = \{1,2,5,6,9,10,12,15,16,18\},$$
$$J_2 = \{1,2,7,9,10,12,15,16,17\},$$
$$J_3 = \{0,2,4,5,6,7,9,11,12,18\},$$
$$J_4 = \{0,2,3,4,5,9,11,12,14,15,17,18\}.$$

(d) $4-(93;46,45,47,56;101)$ sds of type $(3,1,-1,-19)$, with S_1 of skew type. The sets J_1, J_2, J_4 as in (c), $J_3 = \{1,2,3,4,5,6,7,8,16,18,19\}$.

Case $n = 121$. $H = \{1,3,9,27,81\}$ is the subgroup of G of order 5. The orbits α_{2i} are:

$$\alpha_0 = H,\ \alpha_2 = 2H,\ \alpha_4 = 4H,\ \alpha_6 = 5H, \quad \alpha_8 = 7H,\ \alpha_{10} = 8H,\ \alpha_{12} = 10H,$$
$$\alpha_{14} = 11H,\ \alpha_{16} = 16H,\ \alpha_{18} = 17H, \quad \alpha_{20} = 19H,\ \alpha_{22} = 20H.$$

The first four quadruples define nonequivalent $4-(121;60,55,60,70;124)$ sds's of type $(11,1,1,-19)$, with S_1 of skew type.

(a) $J_1 = \{0,2,4,7,8,11,13,14,16,19,20,22\}$,

$$J_2 = \{0,1,4,5,8,9,10,15,17,20,23\},$$
$$J_3 = \{1,2,3,7,9,16,18,19,20,21,22,23\},$$
$$J_4 = \{0,2,9,10,11,12,13,14,15,17,18,21,22,23\}.$$

(b) J_1, J_3, J_4 as in (a), $J_2 = \{0, 1, 4, 5, 8, 9, 11, 14, 16, 21, 22\}$.

(c) J_1, J_2, J_3 as in (a), $J_4 = \{1, 3, 8, 10, 11, 12, 13, 14, 15, 16, 19, 20, 22, 23\}$.

(d) The sets J_1 and J_3 are the same as in (a), J_2 as in (b), and J_4 as in (c).

(e) $4-(121; 55, 55, 55, 55; 99)$ sds of type $(11, 11, 11, 11)$.

$$J_1 = \{0, 1, 2, 4, 8, 9, 11, 12, 14, 20, 23\},$$
$$J_2 = \{1, 2, 3, 6, 7, 11, 13, 17, 18, 19, 20\},$$
$$J_3 = \{2, 5, 7, 10, 13, 14, 17, 20, 21, 22, 23\},$$
$$J_4 = \{2, 5, 12, 13, 16, 17, 18, 19, 21, 22, 23\}.$$

Case $n = 129$. $H = \{1, 4, 16, 64, 97, 121, 127\}$ is the subgroup of G of order 7. The orbits α_{2i} are:

$$\alpha_0 = H, \quad \alpha_2 = 3H, \quad \alpha_4 = 5H, \quad \alpha_6 = 7H, \quad \alpha_8 = 9H, \quad \alpha_{10} = 11H,$$
$$\alpha_{12} = 13H, \quad \alpha_{14} = 19H, \quad \alpha_{16} = 21H, \quad \alpha_{18} = \{43\}.$$

(a) $4-(129; 64, 57, 58, 70; 120)$ sds of type $(15, 13, 1, -11)$, with S_1 of skew type.

$$J_1 = \{1, 2, 4, 7, 9, 11, 12, 14, 16, 18\},$$
$$J_2 = \{0, 1, 2, 3, 9, 11, 14, 15, 19\},$$
$$J_3 = \{0, 1, 3, 6, 8, 10, 12, 16, 18, 19\},$$
$$J_4 = \{0, 3, 7, 8, 9, 10, 12, 14, 15, 17\}.$$

(b) $4-(129; 64, 57, 71, 70; 133)$ sds of type $(15, 1, -11, -13)$, with S_1 of skew type. J_1, J_2, J_4 as in (a), $J_3 = \{1, 2, 3, 4, 6, 8, 9, 11, 14, 15, 18\}$.

Case $n = 133$. $H = \{1, 4, 16, 25, 64, 93, 100, 106, 123\}$ is a cyclic subgroups of G of order 9. The orbits α_{2i} are:

$$\alpha_0 = H, \quad \alpha_2 = 2H, \quad \alpha_4 = 3H, \quad \alpha_6 = 6H, \quad \alpha_8 = 7H,$$
$$\alpha_{10} = 9H, \quad \alpha_{12} = 18H, \quad \alpha_{14} = \{19, 38, 76\}.$$

The first two quadruples define nonequivalent $4-(133; 66, 66, 66, 78; 143)$ sds's of type $(1, 1, 1, -23)$, with S_1 of skew type.

(a) $J_1 = \{1, 2, 5, 6, 9, 11, 12, 14\}$, $J_2 = \{1, 4, 7, 9, 10, 12, 13, 15\}$,
$J_3 = \{0, 5, 6, 8, 11, 12, 13, 15\}$, $J_4 = \{0, 1, 2, 5, 7, 8, 9, 13, 14, 15\}$.

(b) $J_1 = \{0, 2, 4, 7, 9, 11, 13, 14\}$, $J_2 = \{0, 1, 6, 7, 9, 11, 12, 15\}$,
$J_3 = \{0, 2, 3, 5, 6, 8, 11, 15\}$, $J_4 = \{0, 1, 5, 8, 9, 10, 11, 12, 14, 15\}$.

Let S_1, S_2, S_3, S_4 be one of these two sds's and A_1, A_2, A_3, A_4 the corresponding circulant matrices of order 133. By replacing A_1 and A_4 with $-A_4$ and A_1, respectively, in (2), we obtain a Hadamard matrix of order 532 whose excess (sum of all its entries) is $23 \cdot 532 = 12236$. Hence this HADAMARD matrix has maximal possible excess among all HADAMARD matrices of order 532, see [**4**].

The next two quadruples define nonequivalent $4-(133; 66, 72, 72, 75; 152)$ sds's of type $(1, -11, -11, -17)$, with S_1 of skew type.

(c) J_1 as in (a), $J_2 = \{2, 3, 7, 8, 9, 11, 12, 13\}$,
$J_3 = \{0, 1, 5, 6, 9, 10, 11, 13\}$, $J_4 = \{0, 3, 4, 5, 6, 7, 9, 13, 14\}$.

(d) J_1 as in (a), J_2 as in (c), $J_3 = \{0, 1, 4, 7, 8, 10, 11, 12\}$,
$J_4 = \{1, 2, 4, 5, 6, 7, 8, 12, 15\}$.

The next two quadruples define nonequivalent $4-(133; 66, 57, 63, 72; 125)$ sds's of type $(19, 7, 1, -11)$, with S_1 of skew type.

(e) $J_1 = \{0, 2, 5, 6, 8, 11, 13, 14\}$, $J_2 = \{0, 3, 4, 6, 8, 11, 15\}$,
$J_3 = \{0, 1, 4, 7, 9, 10, 11\}$, $J_4 = \{0, 1, 3, 5, 6, 7, 9, 12\}$.

(f) J_1 as in (b), $J_2 = \{2, 3, 5, 6, 8, 12, 13\}$,
$J_3 = \{0, 3, 5, 7, 8, 12, 14\}$, $J_4 = \{0, 2, 3, 4, 5, 6, 9, 11\}$.

The last three quadruples define nonequivalent $4 - (133; 57, 69, 69, 71; 134)$ sds's of type $(19, -5, -5, -11)$.

(g) $J_1 = \{2, 3, 7, 9, 12, 13, 14\}$, $J_2 = \{0, 1, 3, 5, 7, 8, 13, 14, 15\}$,
$J_3 = \{5, 6, 7, 8, 9, 10, 13, 14, 15\}$, $J_4 = \{0, 4, 5, 6, 7, 9, 11, 13\}$.

(h) $J_1 = \{0, 1, 7, 9, 10, 12, 14\}$, $J_2 = \{1, 2, 5, 7, 9, 11, 13, 14, 15\}$,
$J_3 = \{1, 3, 5, 6, 8, 10, 12, 14, 15\}$, $J_4 = \{0, 1, 4, 5, 6, 7, 9, 10\}$.

(i) J_3 and J_4 as in (h), $J_1 = \{0, 1, 6, 8, 11, 13, 15\}$,
$J_2 = \{0, 3, 4, 6, 8, 10, 12, 14, 15\}$.

Case $n = 217$. $H = \{1, 8, 9, 25, 51, 64, 72, 78, 81, 142, 190, 191, 193, 200, 214\}$ is a subgroup of G of order 15. The orbits α_{2i} are:

$$\alpha_0 = H, \quad \alpha_2 = 2H, \quad \alpha_4 = 4H, \quad \alpha_6 = 5H, \quad \alpha_8 = 7H,$$
$$\alpha_{10} = 10H, \quad \alpha_{12} = 19H, \quad \alpha_{14} = \{31, 62, 124\}.$$

The quadruples below define nonequivalent $4 - (217; 108, 108, 111, 123; 233)$ sds's of type $(1, 1, -5, -29)$. In the first two cases S_1 is of skew type.

(a) $J_1 = \{0, 3, 5, 7, 8, 11, 12, 14\}$, $J_2 = \{1, 3, 4, 7, 9, 11, 12, 15\}$,
$J_3 = \{3, 4, 5, 6, 7, 9, 10, 14, 15\}$, $J_4 = \{1, 3, 4, 5, 7, 8, 11, 13, 14\}$.

(b) J_1 as in (a), $J_2 = \{0, 2, 5, 6, 8, 10, 13, 14\}$,
$J_3 = \{2, 4, 5, 6, 7, 8, 11, 14, 15\}$, $J_4 = \{0, 2, 4, 5, 6, 9, 10, 12, 15\}$.

(c) $J_1 = \{0, 2, 3, 6, 9, 11, 12, 15\}$, $J_2 = \{2, 3, 5, 6, 9, 11, 12, 15\}$,
$J_3 = \{1, 4, 5, 6, 7, 9, 13, 14, 15\}$, $J_4 = \{0, 1, 3, 5, 6, 9, 11, 13, 14\}$.

(d) $J_1 = \{0, 5, 6, 9, 10, 11, 12, 15\}$, $J_2 = \{1, 4, 5, 7, 8, 10, 13, 14\}$,
$J_3 = \{4, 6, 7, 8, 10, 11, 12, 14, 15\}$, $J_4 = \{0, 2, 3, 4, 6, 8, 10, 11, 15\}$.

Case $n = 219$. $H = \{1, 4, 16, 37, 55, 64, 148, 154, 178\}$ is the subgroup of G of order 9. The orbits α_{2i} are:

$$\alpha_0 = H,\ \alpha_2 = 2H,\ \alpha_4 = 3H,\ \alpha_6 = 5H,\ \alpha_8 = 7H,\ \alpha_{10} = 9H,\ \alpha_{12} = 11H,$$
$$\alpha_{14} = 15H,\ \alpha_{16} = 19H,\ \alpha_{18} = 22H,\ \alpha_{20} = 23H,\ \alpha_{22} = 33H,\ \alpha_{24} = \{73\}.$$

The first three quadruples define nonequivalent $4-(219; 109, 100, 101, 117; 208)$ sds's of type $(19, 17, 1, -15)$. In the first two cases S_1 is of skew type.

(a) $J_1 = \{1, 3, 5, 6, 8, 11, 12, 15, 17, 18, 21, 22, 24\}$,
$J_2 = \{2, 6, 8, 10, 11, 12, 13, 16, 19, 22, 23, 24\}$,
$J_3 = \{0, 1, 5, 6, 10, 11, 13, 14, 17, 20, 21, 24, 25\}$,
$J_4 = \{0, 2, 3, 4, 5, 6, 7, 11, 12, 13, 16, 20, 23\}$.

(b) J_1, J_2, J_3 as in (a), $J_4 = \{1, 2, 3, 4, 5, 6, 7, 10, 12, 13, 17, 21, 22\}$.

(c) $J_1 = \{2, 3, 5, 6, 10, 13, 14, 15, 18, 19, 20, 21, 24\}$,
$J_2 = \{1, 2, 3, 8, 9, 10, 11, 13, 14, 16, 22, 25\}$,
$J_3 = \{0, 1, 3, 5, 6, 10, 11, 14, 16, 19, 21, 24, 25\}$,
$J_4 = \{1, 3, 6, 9, 10, 11, 12, 13, 17, 18, 21, 22, 23\}$.

(d) $4-(219; 100, 101, 110, 117; 209)$ sds of type $(19, 17, 1, -15)$.

$J_1 = \{0, 2, 5, 8, 9, 10, 11, 13, 14, 19, 21, 25\}$,
$J_2 = \{0, 4, 5, 6, 7, 12, 13, 14, 15, 17, 19, 24, 25\}$,
$J_3 = \{0, 2, 4, 6, 7, 8, 14, 15, 16, 17, 20, 22, 24, 25\}$,
$J_4 = \{0, 1, 2, 3, 5, 6, 10, 12, 13, 18, 21, 22, 23\}$.

(e) $4-(219; 100, 109, 117, 118; 225)$ sds of type $(19, 1, -15, -17)$.

$J_1 = \{2, 3, 4, 5, 7, 12, 13, 14, 15, 17, 18, 25\}$,
$J_2 = \{0, 1, 2, 4, 5, 6, 8, 13, 15, 17, 18, 23, 25\}$,
$J_3 = \{0, 3, 5, 6, 11, 12, 13, 17, 18, 19, 21, 22, 23\}$,
$J_4 = \{1, 2, 3, 5, 6, 7, 10, 14, 15, 16, 17, 19, 21, 24\}$.

(f) $4-(219; 100, 110, 117, 118; 226)$ sds of type $(19, -1, -15, -17)$.

$J_1 = \{2, 6, 8, 10, 11, 13, 14, 17, 18, 20, 23, 24\}$,
$J_2 = \{0, 3, 4, 5, 6, 8, 9, 10, 13, 15, 17, 19, 24, 25\}$,
$J_3 = \{0, 3, 5, 8, 9, 12, 14, 16, 17, 20, 21, 22, 23\}$,
$J_4 = \{0, 3, 5, 6, 7, 8, 9, 10, 11, 12, 18, 20, 23, 25\}$.

(g) $4-(219;101,109,117,119;227)$ sds of type $(17,1,-15,-19)$.

$J_1=\{4,5,6,7,10,12,13,15,17,20,21,24,25\}$,
$J_2=\{2,5,8,10,12,13,17,18,19,21,22,23,24\}$,
$J_3=\{1,2,3,9,10,12,13,15,16,19,21,22,23\}$,
$J_4=\{1,2,3,4,5,7,10,14,16,17,19,21,23,24,25\}$.

(h) $4-(219;101,110,117,119;228)$ sds of type $(17,-1,-15,19)$.

$J_1=\{2,3,10,12,13,14,15,16,17,18,23,24,25\}$,
$J_2=\{0,6,7,8,10,11,12,15,16,19,20,23,24,25\}$,
$J_3=\{1,2,5,8,9,11,13,15,17,18,21,22,23\}$,
$J_4=\{0,1,4,8,10,11,12,13,14,15,17,19,21,24,25\}$.

(i) $4-(219;109,117,118,119;244)$ sds of type $(1,-15,-17,-19)$.

$J_1=\{0,2,3,5,6,10,13,14,15,16,17,19,25\}$,
$J_2=\{1,4,6,7,8,9,11,12,13,20,21,22,23\}$,
$J_3=\{1,4,5,8,9,10,13,17,18,19,20,21,22,24\}$,
$J_4=\{1,3,8,9,10,11,14,15,17,18,19,21,22,24,25\}$.

(j) $4-(219;110,117,118,119;245)$ sds of type $(-1,-15,-17,-19)$.

$J_1=\{1,3,4,5,7,8,10,12,16,17,19,22,24,25\}$,
$J_2=\{1,2,3,4,9,10,12,13,16,20,21,22,23\}$,
$J_3=\{0,1,4,5,9,12,13,14,15,18,19,21,23,25\}$,
$J_4=\{0,1,2,3,4,8,11,12,13,14,15,17,23,24,25\}$.

The next four quadruples define nonequivalent $4-(219;101,101,108,118;209)$ sds's of type $(17,17,3,-17)$.

$J_1=\{0,2,3,5,8,11,14,15,18,19,20,24,25\}$,
$J_2=\{0,2,8,10,11,14,15,16,17,18,20,24,25\}$,
$J_3=\{1,6,7,8,11,13,15,17,20,21,22,23\}$,
$J_4=\{0,2,3,4,5,12,14,16,18,19,20,22,23,24\}$.

(l) J_1, J_2, J_4 as in (k), $J_3=\{0,6,7,9,10,12,14,16,20,21,22,23\}$.

(m) J_2, J_3, J_4 as in (k), $J_1=\{1,2,3,4,9,10,14,15,18,19,21,24,25\}$.

(n) J_1 as in (m), J_2 and J_4 as in (k), and J_3 as in (l).

The quadruples (o) and (p) below define $4-(219;101,108,118,118;226)$ sds's of type $(17,3,-17,-17)$.

(o) $J_1=\{0,1,5,6,8,10,11,12,14,16,21,24,25\}$,
$J_2=\{0,1,4,9,13,15,16,17,19,21,22,23\}$,
$J_3=\{0,1,3,4,5,7,9,11,12,18,19,20,22,24\}$,
$J_4=\{3,4,5,6,8,10,12,13,16,17,18,20,22,24\}$.

(p) $J_1 = \{1,3,4,6,7,15,17,19,21,22,23,24,25\}$,
$J_2 = \{0,6,7,8,10,11,12,17,19,21,22,23\}$,
$J_3 = \{0,2,5,6,7,8,10,11,14,16,17,19,23,24\}$,
$J_4 = \{2,3,4,6,8,10,14,15,18,19,20,23,24\}$.

Case $n = 267$. $H = \{1,4,16,64,67,91,97,121,217,223,256\}$ is the subgroup of G of order 11. The orbits α_{2i} are:

$$\alpha_0 = H,\ \alpha_2 = 2H,\ \alpha_4 = 3H,\ \alpha_6 = 5H,\ \alpha_8 = 7H,\ \alpha_{10} = 9H,\ \alpha_{12} = 10H,$$
$$\alpha_{14} = 13H,\ \alpha_{16} = 14H,\ \alpha_{18} = 15H,\ \alpha_{20} = 19H,\ \alpha_{22} = 39H,\ \alpha_{24} = \{89\}.$$

The first two quadruples define nonequivalent $4-(267;133,121,133,144;264)$ sds's of type $(25,1,1,-21)$, with S_1 of skew type.

(a) $J_1 = \{0,3,4,7,8,11,13,15,16,19,21,22,25\}$,
$J_2 = \{0,1,4,5,6,8,14,15,18,21,23\}$,
$J_3 = \{0,2,4,5,7,9,10,11,14,15,16,17,25\}$,
$J_4 = \{0,1,3,4,6,14,15,16,17,18,20,22,23,25\}$.

(b) J_2 and J_4 as in (a), $J_1 = \{1,2,5,6,9,10,12,14,17,18,20,23,24\}$,
$J_3 = \{1,3,4,5,6,8,10,11,14,15,16,17,24\}$.

The next two quadruples define nonequivalent $4-(267;121,123,133,133;243)$ sds's of type $(25,21,1,1)$.

(c) $J_1 = \{0,1,5,7,8,9,10,11,13,20,23\}$,
$J_2 = \{1,2,3,4,5,8,11,16,17,20,21,24,25\}$,
$J_3 = \{0,2,3,5,6,7,15,17,18,21,22,23,25\}$,
$J_4 = \{1,2,6,7,8,11,12,18,19,20,21,22,25\}$.

(d) J_1, J_3, J_4 as in (c), $J_2 = \{0,2,3,4,5,9,10,16,17,20,21,24,25\}$.

The next four quadruples define nonequivalent $4-(267;133,122,122,132;242)$ sds's of type $(23,23,3,1)$, with S_1 of skew type.

(e) $J_1 = \{0,2,4,7,9,10,13,15,17,18,20,23,24\}$,
$J_2 = \{1,3,7,9,11,13,16,17,20,22,23,25\}$,
$J_3 = \{1,6,7,10,11,14,17,19,20,21,23,25\}$,
$J_4 = \{2,3,8,9,11,14,15,17,18,21,22,23\}$.

(f) J_1, J_2, J_4 as in (e), $J_3 = \{2,6,8,10,12,13,14,19,20,21,22,25\}$.

(g) J_1, J_3, J_4 as in (e), $J_2 = \{0,1,2,4,5,8,11,12,13,14,22,25\}$.

(h) J_1 and J_4 as in (e), J_2 as in (g), and J_3 as in (f).

This completes the proof.

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