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ON A LINEAR FUNCTIONAL-DIFFERENTIAL EQUATION

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The functional-differential equation (1) is treated by an ad hoc method to obtain solutions involving one or more arbitrary functions, and also solutions which satisfy some additional conditions.

There are many results concerning the linear functional-differential equation for the unknown functions $u : S \to \mathbb{R}$

$$u'(f(x)) + au(f(x)) + bu'(x) + cu(x) = 0,$$

where S is a nonempty set, $f : S \to S$ and $a, b, c \in \mathbb{R}$ are given, particularly if there exists a positive integer n such that $f^n(x) = x$. In this note we shall consider an equation of the same type, but the unknown function will be in two variables.

Let S be a nonempty set and let $f : S \to S$ be a given mapping. Consider the equation in u:

$$u_y(f(x), y) + au(f(x), y) + bu_y(x, y) + cu(x, y) = 0, \tag{1}$$

where $a, b, c \in \mathbb{R}$ are given and $u : S \times \mathbb{R} \to \mathbb{R}$ is the unknown function with $u_y(x, y) = \dfrac{\partial u(x, y)}{\partial y}$.

This equation can be written in the form

$$(L_1 L_2 + aL_1 + bL_2 + cI)u = 0, \tag{2}$$

where the linear operators L_1, L_2, I are defined by

$$L_1 u(x, y) = u(f(x), y), \qquad L_2 u(x, y) = u_y(x, y), \qquad Iu(x, y) = u(x, y)$$

We shall apply to (2) a variant of the method suggested in [**1**] which is based upon the existence of characteristic vectors for both operators L_1, L_2, i.e. upon the existence of functions u with the properties

$$L_1 u = \lambda u, \qquad L_2 u = \mu u,$$

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i.e.

$$u(f(x),y)=\lambda u(x,y),\qquad u_y(x,y)=\mu u(x,y). \tag{3}$$

From the second equation in (3) follows $u(x,y)=\varphi(x)e^{\mu y}$, where φ is an arbitrary function. Substituting this into the first equation of (3) we arrive at the functional equation

$$\varphi(f(x))=\lambda\varphi(x). \tag{4}$$

Equation (4) is the so-called Schröder equation (or the Schröder-Koenigs equation). This equation was thoroughly investigated in many papers; see, for example, Chapter 6 of [**2**].

In further text v_λ will denote an arbitrary solution of (4): it may be trivial. Therefore, the function u defined by

$$u(x,y)=v_\lambda(x)e^{\mu y} \tag{5}$$

satisfies the conditions (3) and substituting (5) into (2) we obtain the characteristic equation

$$\lambda\mu+a\lambda+b\mu+c=0 \tag{6}$$

which implies

$$\mu=-\frac{a\lambda+c}{\lambda+b}$$

and (5) becomes

$$u(x,y)=v_\lambda(x)e^{-\frac{a\lambda+c}{\lambda+b}y}.$$

Since the equation (2) is linear, we conclude that it has the following formal solution

$$u(x,y)=\sum_\lambda v_\lambda(x)e^{-\frac{a\lambda+c}{\lambda+b}y}, \tag{7}$$

where the sum is taken over all $\lambda\in\Lambda\subset\mathbb{R}$ on supposition that it exists. Also, from (6) follows

$$\lambda=-\frac{b\mu+c}{\mu+a}$$

and (2) has the following formal solution

$$u(x,y)=\sum_\mu v_{-\frac{b\mu+c}{\mu+a}}(x)e^{\mu y}, \tag{8}$$

where the sum has a similar meaning as the sum in (7). Combining (7) and (8) we obtain the following solution of (2):

$$u(x,y)=\sum_\lambda v_\lambda(x)e^{-\frac{a\lambda+c}{\lambda+b}y}+\sum_\mu v_{-\frac{b\mu+c}{\mu+a}}(x)e^{\mu y}. \tag{9}$$

The obtained solution (9) may seem to be rather formal and therefore useless. However, this is not so, since in certain cases explicit solutions can be derived from (9).

First of all, if $c = ab$, then (9) becomes

$$u(x,y) = e^{-ay}\sum_{\lambda} v_\lambda(x) + v_{-b}(x)\sum_{\mu} e^{\mu y}. \tag{10}$$

Then putting

$$\sum_{\lambda} v_\lambda(x) = F(x), \qquad \sum_{\mu} e^{\mu y} = G(y),$$

we get

$$u(x,y) = e^{-ay}F(x) + v_{-b}(x)G(y). \tag{11}$$

This suggests that (11), where F is an arbitrary and G an arbitrary differentiable function, might be a solution of (2). A direct verification show that this is indeed true.

Furthermore, for certain choices of f we can get an explicit expression for $v_\lambda(x)$. For instance, if

(i) $f(x) = x + d$, where $d \neq 0$, then

$$v_\lambda(x) = P(x)\lambda^{x/d},$$

where P is an arbitrary periodic function with period d; if

(ii) $f(x) = dx$, where $d > 0$, then

$$v_\lambda(x) = P(\log_d x)\lambda^{\log_d x},$$

where P is an arbitrary periodic function with period 1.

On the other hand, if there exists a positive integer n such that $f^n(x) = x$, nontrivial solutions of (4) exist if and only if $\lambda^n = 1$. So, for example, if

(iii) $f(x) = 1 - x$ (the case $n = 2$), then

$$v_1(x) = F(x) + F(1-x), \qquad v_{-1}(x) = F(x) - F(1-x),$$

where F is an arbitrary function and $v_\lambda(x) = 0$ for $\lambda^2 \neq 1$; if

(iv) $f(x) = \dfrac{x-1}{x}$ (the case $n = 3$), then

$$v_1(x) = F(x) + F\Big(\frac{x-1}{x}\Big) + F\Big(\frac{1}{1-x}\Big),$$

where F is an arbitrary function and $v_\lambda(x) = 0$ for $\lambda \neq 1$.

The following examples will provide further illustrations how the formal solution (9) can be used in solving concrete problems.

Example 1. If $A, B : \mathbb{R} \to \mathbb{R}$ are given functions such that $A(0) = B(0) = \alpha$, consider the Goursat type problem

$$u_y(x+1, y) + 2u(x+1, y) - 3u_y(x, y) - 6u(x, y) = 0, \tag{12}$$
$$u(x, 0) = A(x), \qquad u(0, y) = B(y). \tag{13}$$

In this case $c = ab$, and using (11) we obtain the following solution of (12):

$$u(x, y) = e^{2y} F(x) + v_3(x) G(y),$$

where F is an arbitrary and G an arbitrary differentiable function. Furthermore, the equation (4) becomes $\varphi(x+1) = 3\varphi(x)$ and so

$$v_3(x) = 3^x P(x) \qquad (P \text{ arbitrary 1-periodic function}).$$

Hence,

$$u(x, y) = e^{2y} F(x) + 3^x P(x) G(y). \tag{14}$$

Substituting the conditions (13) into (14) we easily obtain the following solution of the problem (12)–(13):

$$u(x, y) = e^{2y} A(x) + \frac{1}{P(0)} \big(3^x P(x) B(y) - \alpha 3^x e^{2y} P(x)\big).$$

Example 2. For the equation

$$u_y(3-x, y) - 2u(3-x, y) + 2u_y(x, y) - 4u(x, y) = 0 \tag{15}$$

the corresponding equation (4)

$$\varphi(3-x) = -2\varphi(x)$$

has no nontrivial solutions, and according to (11) we obtain the following solution of (15):

$$u(x, y) = e^{2y} F(x) \qquad (F \text{ arbitrary}).$$

On the other hand, for the equation

$$u_y(1-x, y) - 2u(1-x, y) - u_y(x, y) + 2u(x, y) = 0 \tag{16}$$

the equation (4) becomes

$$\varphi(1-x) = \varphi(x)$$

and so

$$v_1(x) = H(1-x) + H(x),$$

where H is an arbitrary function. Hence, we obtain the following solution of (16):

$$u(x, y) = \big(H(1-x) + H(x)\big) F(y) + e^{2y} G(x),$$

where F is an arbitrary differentiable function and G, H are arbitrary functions.

Remark. Putting $H(x)F(y) = T(x,y)$, it is tempting to try whether

$$u(x,y) = T(1-x,y) + T(x,y) + e^{2y}G(x)$$

where T is an arbitrary function, differentiable in y, is also a solution of (16). A direct verification shows that this is true.

Example 3. Consider the problem consisting of the equation

$$u_y(x+1,y) + u(x+1,y) - 3u_y(x,y) + 4u(x,y) = 0, \tag{17}$$

and the condition

$$u(0,y) = \operatorname{sh} 2y. \tag{18}$$

Since $v_\lambda(x) = \lambda^x P_\lambda(x)$, where P_λ is an arbitrary 1-periodic function, we obtain the following solution of (17):

$$u(x,y) = \sum_\lambda \lambda^x P_\lambda(x) e^{-\frac{\lambda+4}{\lambda-3}y}, \tag{19}$$

since the second sum in (9) will contribute nothing new. From the equations $-\dfrac{\lambda+4}{\lambda-3} = \pm 2$ we get $\lambda = \dfrac{2}{3}$ and $\lambda = 10$, respectively. Hence, we take

$$P_{2/3}(x) = \frac{1}{2}, \quad P_{10}(x) = -\frac{1}{2}, \quad P_\lambda(x) = 0 \text{ for other values of } \lambda.$$

Therefore, (19) becomes

$$u(x,y) = \frac{1}{2}\Big(\frac{2}{3}\Big)^x e^{2y} - \frac{1}{2}(10)^x e^{-2y},$$

and this is a solution of (17)–(18).

Example 4. A formal solution of the equation

$$u_y\Big(\frac{x}{2x-1}, y\Big) + 2u\Big(\frac{x}{2x-1}, y\Big) - 2u_y(x,y) + 3u(x,y) = 0 \tag{20}$$

is given by

$$u(x,y) = \sum_\lambda v_\lambda(x) e^{\frac{2\lambda+3}{2-\lambda}y}.$$

However, the corresponding equation (4)

$$\varphi\Big(\frac{x}{2x-1}\Big) = \lambda\varphi(x)$$

has nontrivial solutions only for $\lambda = 1$ and $\lambda = -1$. They are

$$v_1(x) = F(x) + F\Big(\frac{x}{2x-1}\Big), \qquad v_{-1}(x) = G(x) - G\Big(\frac{x}{2x-1}\Big),$$

where F, G are arbitrary functions. Therefore, we obtain the following solution of (20):

$$u(x,y)=\left(F(x)+F\Big(\frac{x}{2x-1}\Big)\right)e^{5y}+\left(G(x)-G\Big(\frac{x}{2x-1}\Big)\right)e^{y/3}.$$

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