Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 2 (1991), 82–88.

SOME RESULTS ON THE REDUCED ENERGY OF GRAPHS, III

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In a recent paper [10], A. Torgašev described all finite connected graphs whose energy (i.e. the sum of all positive eigenvalues including also their multiplicities), does not exceed 3. In this paper, we describe all connected graphs whose reduced energy, i.e. the sum of absolute values of all eigenvalues except the least and the largest ones, does not exceed 2.5.

In this paper we consider only finite connected graphs having no loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$, and its order (number of vertices) by $|G|$. The spectrum of such a graph is the family $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of eigenvalues of its 0–1 adjacency matrix, and we also write $\lambda_i(G) = \lambda_i$, $(i = 1, 2, \ldots, n)$. The eigenvalue $\lambda_1(G) = r(G)$ is called the spectral radius of G, while the eigenvalue $\lambda_n(G)$ is the least eigenvalue of G.

The sum of eigenvalues $|\lambda_1| + |\lambda_2| + \cdots + |\lambda_{n-1}|$ is denoted by $S_1(G)$ and has been investigated in [**6**]. All connected graphs with the property $S_1(G) \leq 6$ has been determined in [**6**].

Next, the sum of eigenvalues $|\lambda_2| + |\lambda_3| + \cdots + |\lambda_{n-1}|$ is denoted by $T_1(G)$ and called the *reduced energy* of G. For any real $a > 0$, we can consider the class of graphs

$$E_1(a) = \big\{G \mid T_1(G) \leq a\big\}.$$

In this paper we completely describe the class $E_1(2.5)$.

Briefly, any graph $G \in E_1(2.5)$ is called *admissible,* and any other graph *impossible* (or forbidden) for this class.

If H is any connected (induced) subgraph of a graph G, we write $H \subseteq G$. Making use of the known interlacing theorem [**1**, p. 19] we have $T_1(H) \leq T_1(G)$. Whence, we have that any connected subgraph of an admissible graph is also ad-

1991 Mathematics Subject Classification: 05C50 missible. This implies that the method of forbidden subgraphs can be consistently applied.

Next, let $K_{n_1,n_2,\ldots,n_m}$, P_n and C_n be the complete m-partite graph, the path and the cycle with n vertices, respectively. Since the complete m-partite graph $K_{n_1,n_2,\ldots,n_m}$ has just one positive eigenvalue $\lambda_1(G)$, it will belong to the class $E_1(a)$ if and only if $\lambda_1(G) + \lambda_n(G) \leq a$.

Since the graph $K_{m,n}$ belongs to the class $E_1(a)$ for every $m, n \in \mathrm{N}$ we conclude that class $E_1(a)$ is infinite for every constant $a > 0$.

In order to generate all graphs from the class $E_1(2.5)$, we firstly determine the complete set of the so-called canonical graphs in this class.

We say that two vertices $x, y \in V(G)$ are equivalent in G and denote it by $x \sim y$ if x is nonadjacent to y, and x and y have exactly the same neighbors in G. Relation $\sim$ is obviously an equivalence relation on the vertex set $V(G)$. The corresponding quotient graph is denoted by g, and called the *canonical* graph of G. The last graph is also connected, and we obviously have $g \subseteq G$. For instance, if $G = K_{m_1,m_2,\ldots,m_p}$ $(p \geq 2)$ is the complete p-partite graph, then its canonical graph is the complete graph K_p. The canonical graph of the complete graph K_n is the same graph K_n.

We say that G is canonical if $|G| = |g|$, thus if G has no two equivalent vertices.

Let g be the canonical graph of G, $|g| = k$, and $N_1, \ldots, N_k$ be the corresponding sets of equivalent vertices in G. Then we denote $G = g(N_1, \ldots, N_k)$, or simply $G = g(n_1, \ldots, n_k)$, where $|N_i| = n_i$ $(i = 1, \ldots, k)$, understanding that g is a labelled graph. We call $N_1, \ldots, N_k$ the characteristic sets of G. Obviously, each set $N_i \subseteq V(G)$ $(i = 1, \ldots, k)$ consists only of isolated vertices, and if at least one edge between the sets N_i, N_j $(i \neq j)$ is present, then all possible edges between these sets are also present.

If g is the canonical graph of a graph G, we have that $g \subseteq G$ whence we obtain

$$G \in E_1(a) \Rightarrow g \in E_1(a).$$

Hence, it is very convenient to describe firstly the set of all canonical graphs from the set $E_1(a)$.

We note that many other hereditary problems in the spectral theory of graphs can be reduced to finding firstly the corresponding sets of canonical graphs. In this respect one can consult the papers [**4**], [**7**], [**10**] etc.

Creating the complete set of canonical graphs in this paper is based on the following general theorem proved in [**11**], which can be very valuable for other similar problems.

Theorem A. *In all but a sequence of exceptional cases, each connected canonical graph on n vertices $(n \geq 3)$ contains an induced subgraph on $n-1$ vertices, which is also connected and canonical. The mentioned exceptional cases are the graphs in Fig. 1. (In graphs in Fig. 1 vertices y_i and x_j are adjacent whenever $i \leq j$).*

Fig. 1.

The above exceptional graphs satisfy the relations $T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$.

Now, we prove an important property of the general class $E_1(a)$ $(a > 0)$. It is based on the Theorem B which is proved in [**9**].

Theorem B. *For every $n \in N$ the complete set of canonical graphs which have n nonzero eigenvalues is finite.*

Theorem 1. *For every constant $a > 0$ the set of canonical graphs from the class $E_1(a)$ is finite.*

Proof. On the contrary, assume that the set of canonical graphs from the class $E_1(a)$ is infinite for some $a > 0$. Then, by Theorem B, for every real number $M > 0$, there exists a graph G such that

$$|\lambda_2| + |\lambda_3| + \cdots + |\lambda_{n-1}| \le a, \tag{1}$$

which has $p \ge M$ nonzero eigenvalues. The multiplicity of zero of the graph G is then $q = n - p$. Assume that $\lambda_s > \lambda_{s+1} = \cdots = \lambda_{s+q} = 0 > \lambda_{s+q+1}$. The corresponding characteristic polynomial of the graph G is then

$$P_n(\lambda) = \lambda^q\big(\lambda^p + a_1\lambda^{p-1} + \cdots + a_p\big),$$

where $|a_p| = \lambda_1\lambda_2\cdots\lambda_s \cdot |\lambda_{s+q+1}|\cdots|\lambda_n|$.

In the sequel, without any loss of generality, we can assume that $q = 0$, thus $n = p$. Besides, we assume that n is chosen so large that we have $\sqrt{n} \ge [a] + 5$. We have that $|\lambda_1|, |\lambda_n| \le n - 1$, while relation (1) gives $|\lambda_i| \le \sqrt{n}$ for $i = 2, 3, \ldots, n-1$.

If $\varepsilon = 1/\sqrt{n}$, let k be the total number of eigenvalues λ_i, with $|\lambda_i| \le \varepsilon$ $(i = 2, 3, \ldots, n-1)$. It is easy to see that $k > [a] + 3$. Indeed, in the contrary case, we would have that there exists at least $\big(n - (k+2)\big)$ eigenvalues λ_i $(2 \le i \le n-1)$ with $|\lambda_i| > \varepsilon$. Relation (1) now gives

$$[a] + 1 > a \ge \sum_{i=2}^{n-1} |\lambda_i| > \frac{n - (k+2)}{\sqrt{n}} > \sqrt{n} - \frac{[a]+5}{\sqrt{n}}. \tag{2}$$

Since $\sqrt{n} \geq [a] + 5$, relation (2) yields $[a] + 1 > \sqrt{n} - 1$, what is a contradiction.

Next, let k_0 be the total number of all eigenvalues λ_i $(i = 2, 3, \ldots, n-1)$ with $|\lambda_i| > 1$. Relation (1) now yields

$$[a] + 1 > a \geq \sum_{i=2}^{n-1} |\lambda_i| \geq \sum_{i=1}^{k_0} 1 = k_0.$$

Whence we get $k_0 \leq [a]$.

Now we finally have

$$|a_n| = |\lambda_1||\lambda_2| \cdots |\lambda_n| = |\lambda_1||\lambda_n|\big(|\lambda_2||\lambda_3| \cdots |\lambda_{n-1}|\big)$$

$$\leq (n-1)^2 \underbrace{\sqrt{n}\sqrt{n} \cdots \sqrt{n}}_{k_0} \cdot \underbrace{\frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} \cdots \frac{1}{\sqrt{n}}}_{k} \cdot \underbrace{1 \cdot 1 \cdot \cdots \cdot 1}_{n-(k+k_0+2)} \leq 1$$

what is a contradiction since $|a_n| \in \mathbf{N}$ $(a_n \neq 0)$. Hence the set $E_1(a)$ is finite for every $a > 0$. ¤

By a direct inspection of spectra of all connected graphs with at most 5 vertices (see, for example, tables in [**1**]), we find that class $E_1(2.5)$ contains exactly 11 canonical graphs with at most 5 vertices. They are displayed in Fig. 2.

Fig. 2.

As is also known, there is exactly 112 nonisomorphic connected graphs with 6 vertices. By a direct inspection of their spectra (see, for example [**3**]), we find that class $E_1(2.5)$ contains no canonical graphs with 6 vertices.

By Theorem A and Theorem 1 we immediately obtain the following result.

Theorem 2. *Figure 2 displays all canonical graphs from the class* $E_1(2.5)$.

Proposition 1. *A graph $G = g_1(m,n) \in E_1(2.5)$ $(m \leq n)$ for all values of parameters m, n.*

Proof. Since $g_1 = K_2$, graph $G = K_{m,n}$ is the complete bipartite graph, hence it has exactly one positive and exactly one negative eigenvalue. Consequently, $T_1(G) = 0$ for every complete bipartite graph G. ¤

Proposition 2. *A graph $G = g_2(m,n,k)$ $(m \leq n \leq k)$ belongs to the class $E_1(2.5)$ if and only if*

$$(m,n,k) = (1,\dot{1},\dot{1}),\ (2,\underset{\cdot}{3},\dot{3}),\ (2,4,4),$$

where $\dot{p}$ means that the corresponding parameter is greater or equal p, and $\underset{\cdot}{p}$ means that the corresponding parameter is less or equal p.

Proof. Since $g_2 = K_3$, graph G is the complete 3-partite graph $K_{m,n,k}$. It has only three nonzero eigenvalues, which are the roots of the polynomial

$$P(\lambda) = \lambda^3 - (mn + mk + nk)\lambda - 2mnk.$$

Therefore $G \in E_1(2.5)$ if and only if $|\lambda_2| \leq 2.5$, that is if and only if $P(-2.5) \geq 0$. Whence we easily find the statement. ¤

Proposition 3. *A graph $G = g_3(m,n,k,l)$ $(m \leq l)$ belongs to the class $E_1(2.5)$ if and only if (m,n,k,l) has one of the following values:*

$$\begin{array}{lll}
(1,\ 1,\ \dot{1},\dot{1}), & (1,\ \dot{1},\ \dot{1},1), & (1,\ \underset{\cdot}{2},\ 1,\underset{\cdot}{8}), \\
(1,\ 2,\ \dot{2},\underset{\cdot}{7}), & (1,\ 3,\ 2,\underset{\cdot}{4}), & (1,\ 3,\ \dot{3},\underset{\cdot}{3}), \\
(1,\ 4,\ \underset{\cdot}{3},\underset{\cdot}{3}), & (1,\ 4,\ \dot{4},2), & (1,\ 5,\ 2,\underset{\cdot}{3}), \\
(1,\ \underset{\cdot}{7},\ \dot{2},2), & (1,\ 8,\underset{\cdot}{26},2), & (1,\ 9,\underset{\cdot}{14},2), \\
(1,10,\underset{\cdot}{10},2), & (1,11,\ \underset{\cdot}{8},2), & (1,12,\ \underset{\cdot}{7},2), \\
(1,\underset{\cdot}{15},\ \underset{\cdot}{6},2), & (1,\underset{\cdot}{21},\ \underset{\cdot}{5},2), & (1,\underset{\cdot}{53},\ \underset{\cdot}{4},2), \\
(1,\dot{54},\ \underset{\cdot}{3},2), & (2,\ 1,\ 1,\underset{\cdot}{5}), & (2,\ 1,\ \underset{\cdot}{3},\underset{\cdot}{4}), \\
(2,\ 1,\ \dot{4},\underset{\cdot}{3}), & (2,\ 2,\ 2,2). &
\end{array}$$

Proof. It is easy to check that each of the above graphs belongs to the class $E_1(2.5)$. Next, note that all eigenvalues of such a graph are determined by equation

$$\lambda^4 - (mn + nk + kl)\lambda^2 + mnkl = 0.$$

Hence, these eigenvalues can be explicitly found. Therefore, it is easy to prove that $G = g_3(m,n,k,l) \in E_1(2.5)$ if and only if

$$256\,mnkl - 400(mn + nk + kl) + 625 \leq 0.$$

Hence, we immediately get the statement. ¤

In a similar way, one can prove the next 8 propositions.

Proposition 4. *A graph $G = g_4(m,n,k,l)$ $(k \le l)$ belongs to the class $E_1(2.5)$ if and only if (m,n,k,l) has one of the following values:*

$$\begin{aligned}(m,n,k,l) = {} & (1,\ 1,\underset{\cdot}{5},\ \dot{1}),\quad (1,\ 1,6,\underset{\cdot}{30}),\quad (1,\ 1,7,\underset{\cdot}{13}),\\ & (1,\ 1,8,\underset{\cdot}{10}),\quad (1,\ 1,9,\ \underset{\cdot}{9}),\quad (1,\ \underset{\cdot}{5},1,\ \dot{1}),\\ & (1,\ \dot{2},\underset{\cdot}{2},\ \underset{\cdot}{2}),\quad (1,\ \underset{\cdot}{6},1,\underset{\cdot}{86}),\quad (1,\ 7,1,\underset{\cdot}{35}),\\ & (1,\ 8,1,\underset{\cdot}{25}),\quad (1,\ 9,1,\underset{\cdot}{21}),\quad (1,10,1,\underset{\cdot}{19}),\\ & (1,11,1,\underset{\cdot}{17}),\quad (1,12,1,\underset{\cdot}{16}),\quad (1,\underset{\cdot}{14},1,\underset{\cdot}{15}),\\ & (1,\underset{\cdot}{17},1,\underset{\cdot}{14}),\quad (1,\underset{\cdot}{24},1,\underset{\cdot}{13}),\quad (1,\underset{\cdot}{43},1,\underset{\cdot}{12}),\\ & (1,\dot{44},1,\underset{\cdot}{11}),\quad (2,\ 1,1,\ \dot{1}),\quad (2,\ 2,1,\ \underset{\cdot}{8}),\\ & (2,\ \underset{\cdot}{5},1,\ \underset{\cdot}{5}),\quad (2,\ \dot{4},1,\ \underset{\cdot}{4}),\quad (3,\ 1,1,\ \underset{\cdot}{6}),\\ & (3,\ \dot{2},1,\ \underset{\cdot}{3}),\quad (4,\ \underset{\cdot}{2},1,\ \underset{\cdot}{3}),\quad (\underset{\cdot}{10},\ \dot{1},1,\ \underset{\cdot}{2}),\\ & (11,\ 1,1,\ \underset{\cdot}{2}),\quad (\dot{12},\ \dot{1},1,\ 1).\end{aligned}$$

Proposition 5. *A graph $G = g_5(m,n,k,l)$ $(m \le n \le k \le l)$ belongs to the class $E_1(2.5)$ if and only if*

$$(m,n,k,l) = (1,1,\underset{\cdot}{2},\dot{1}).$$

Proposition 6. *A graph $G = g_6(m,n,k,l,p)$ $(m \le p)$ belongs to the class $E_1(2.5)$ if and only if (m,n,k,l,p) has one of the following values:*

$$\begin{aligned}&(1,1,1,\dot{1},1),\quad (1,1,1,\underset{\cdot}{2},\underset{\cdot}{2}),\quad (1,1,\dot{1},1,\underset{\cdot}{2})\\ &(1,1,2,\underset{\cdot}{7},1),\quad (1,1,3,\underset{\cdot}{5},1),\quad (1,1,\underset{\cdot}{7},\underset{\cdot}{4},1),\\ &(1,1,\dot{8},\underset{\cdot}{3},1).\end{aligned}$$

Proposition 7. *A graph $G = g_7(m,n,k,l,p)$ $(m \le p)$ belongs to the class $E_1(2.5)$ if and only if (m,n,k,l,p) has one of the following values:*

$$\begin{aligned}&(1,1,1,\underset{\cdot}{2},\underset{\cdot}{5}),\quad (1,1,1,\dot{3},\underset{\cdot}{4}),\quad (1,1,\underset{\cdot}{2},\dot{2},1),\\ &(1,1,\dot{3},1,1),\quad (1,2,1,1,\underset{\cdot}{2}),\quad (1,2,1,\underset{\cdot}{3},1),\\ &(1,2,2,1,1),\quad (1,3,1,\underset{\cdot}{2},1),\quad (1,\dot{3},\underset{\cdot}{2},1,1).\end{aligned}$$

Proposition 8. *A graph $G = g_8(m,n,k,l,p)$ $(m \le n)$ belongs to the class $E_1(2.5)$ if and only if (m,n,k,l,p) has one of the following values:*

$$\begin{aligned}&(1,\ 1,\underset{\cdot}{2},1,\dot{1}),\quad (1,1,\ 3,1,\underset{\cdot}{4}),\quad (1,\ 1,4,1,\underset{\cdot}{3}),\\ &(1,\ 1,\dot{5},1,\underset{\cdot}{2}),\quad (1,2,\ 1,1,\underset{\cdot}{4}),\quad (1,\ 2,2,1,\underset{\cdot}{2}),\\ &(1,\ 2,\dot{3},1,1),\quad (1,3,\ 1,1,\underset{\cdot}{2}),\quad (1,\ 3,\dot{2},1,1),\\ &(1,\ 4,\dot{1},1,1),\quad (1,5,\underset{\cdot}{16},1,1),\quad (1,\ 6,\underset{\cdot}{6},1,1),\\ &(1,\ 7,\underset{\cdot}{4},1,1),\quad (1,\underset{\cdot}{9},\ \underset{\cdot}{3},1,1),\quad (1,\underset{\cdot}{15},\underset{\cdot}{2},1,1),\\ &(1,\dot{16},1,1,1),\quad (2,2,\ 1,1,\underset{\cdot}{2}),\quad (2,\ 2,2,1,1),\\ &(2,\underset{\cdot}{10},1,1,1),\quad (3,\underset{\cdot}{5},\ 1,1,1).\end{aligned}$$

Proposition 9. *A graph* $G = g_9(m,n,k,l,p)$ $(k \le l \le p)$ *belongs to the class* $E_1(2.5)$ *if and only if*

$$(m,n,k,l,p) = (1,\dot{1},1,1,1).$$

Proposition 10. *A graph* $G = g_{10}(m,n,k,l,p)$ $(k \le p)$ *belongs to the class* $E_1(2.5)$ *if and only if*

$$(m,n,k,l,p) = (1,\underset{\cdot}{2},1,1,1).$$

Proposition 11. *A graph* $G = g_{11}(m,n,k,l,p)$ $(m \le n)$ *belongs to the class* $E_1(2.5)$ *if and only if*

$$(m,n,k,l,p) = (1,1,1,1,1).$$

Propositions 1–11 and Theorem 1 completely describe the class $E_1(2.5)$.

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(Received July 8, 1991)