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COPLANAR GRAPHS

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Planar graphs with planar complements are called coplanar. All coplanar graphs are determined by a combined application of mathematical reasoning and computer search.

1. INTRODUCTION

A planar graph with a planar complement is called *coplanar*. Obviously, complement $\bar{G}$ of a coplanar graph G is also coplanar. A self-complementary planar graph is coplanar. Since the complement of a disconnected graph is connected, at least one of the graphs from a complementary coplanar pair is connected.

A fairly general problem, set by F. HARARY, is the following one: *Find the set of all graphs G such that both G and $\bar{G}$ have a given property.* As planarity is one of the important and widely known graph theoretical properties, we decided to find all coplanar graphs. In particular, J. AKIYAMA and F. HARARY [**1**] posed the problem of determining all coplanar graphs. In this paper we shall solve this problem.

In 1984 the authors decided to use the expert system "Graph" [**6**] for finding coplanar graphs. The system "Graph" contains a programming module for planarity testing developed by one of authors (A. J.). The problem was quickly solved in principle but due to some technicalities and travellings of authors it was not completed for a long time. Results are partially announced in conference talks [**4**], [**10**], the thesis [**11**] and the expository article [**12**]. We recently have learnt that in the meantime the problem of coplanar graphs has been solved in [**9**]. The authors of [**9**] have listed all possible degree sequences of coplanar graphs and have used them to construct graphs. They do not mention a possible usage of a computer in their work. We decided to publish our work because of different methods and different results.

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Since the thickness of K_9 is equal to 3 (see [**2**]), a coplanar graph has less than 9 vertices.

The only graph with at most 5 vertices which is not planar is K_5. Therefore all graphs with at most 5 vertices, except for K_5 and $5K_1$, are coplanar.

Since a maximal planar graph on n ($n \geq 3$) vertices has $3n-6$ edges, coplanar graphs on 6 vertices have between 3 and 12 edges, those on 7 vertices have between 6 and 15 edges and those on 8 vertices between 10 and 18 edges.

A coplanar graph is called *lower coplanar* graph if its number of edges does not exceed the number of edges in its complement. Other coplanar graphs are called *upper coplanar* graphs.

It is sufficient to find, or characterize lower coplanar graphs.

Lower coplanar graphs on n vertices for $n = 6, 7, 8$ may have the following numbers of edges

n	minimum	maximum
6	3	7
7	6	10
8	10	14

Maximal planar graphs are called *triangulations* since their faces are triangles. Triangulations are dual graphs of cubic planar 3-connected graphs. Connected cubic graphs up to 14 vertices are given in [**3**] and that is sufficient for the needs of the reported research.

2. CASE $n = 6$

Using tables [**7**] it is simple to conclude that all graphs on 6 vertices are coplanar, but the following 28 graphs: $6K_1$, $K_2 + 4K_1$, $P_3 + 3K_1$, $2K_2 + 2K_1$, $K_{1,3} + 2K_1$, $K_3 + 3K_1$, $P_3 + K_2 + K_1$, $K_{1,4} + K_1$, $K_{1,3} + K_2$, $K_3 + K_2 + K_1$, $2P_3$, $K_3 + P_3$, $K_{1,5}$, $2K_3$, and their complements.

We can characterize lower coplanar graphs on 6 vertices in one more way.

There are only two 3-connected planar cubic graphs on 8 vertices. They are designated by G_1, G_2 in Fig. 1, which also displays their dual graphs H_1, H_2, which themselves are triangulations on 6 vertices. Fig. 1 further shows graphs B_1, B_2 which are complements of H_1, H_2.

Graphs B_1, B_2 are coplanar. Any lower coplanar graph on 6 vertices is obtained from B_1 or B_2 by adding at most 4 edges. No nonplanar graph can be obtained in this way. Hence we have the following proposition.

Proposition 1. *A lower coplanar graph on 6 vertices is any graph on 6 vertices with at most 7 edges which contains a path of length 3 or 3 mutually non-adjacent edges.*

It remains to find coplanar graphs on 7 and 8 vertices.

Fig. 1.

3. CASE $n = 7$

There are 5 planar, 3-connected cubic graphs on 10 vertices [**3**] and they give rise to 5 triangulations on 7 vertices whose complements are displayed in Fig. 2.

Fig. 2.

What is left to be done for the case $n = 7$ is simple. Adding new edges goes up to $n = 10$, otherwise the graph would have more edges than its complement. It can also be easily established that the only nonplanar graphs on 7 vertices with at most 10 edges are those in Fig. 3. Therefore, we state the following theorem.

Theorem 1. *A graph G on 7 vertices is lower coplanar if and only if*

a) *G contains as a spanning subgraph one of the five graphs in Fig. 2, and*

b) *G is not isomorphic to any of the five graphs in Fig. 3.*

It can be readily verified, of course, that each of the graphs of Fig. 3 can be obtained by adding edges to some of the graphs of Fig. 2.

Fig. 3.

A table of connected graphs on 7 vertices from [**5**] was produced by the system "Graph" and we have at our disposition a file with these graphs within the system. Using facilities of the system "Graph", it was easy to extract coplanar graphs. We list below connected lower coplanar graphs, and complements of disconnected ones. Graph identification numbers refer to the table from [**5**].

004										

014	021	023	025	034	035	040				

045	047	049	052	053	054	056	057	058	063	065
068	069	071	073	074	075	076	077	078	079	080
082	083	090	095	098	100	103	104	106	109	

112	114	115	117	118	119	120	122	124	126	127
128	129	130	131	132	133	134	135	136	137	139
140	141	145	146	147	149	150	151	153	155	157
158	159	160	161	162	164	165	166	167	168	169
170	171	172	173	174	175	176	177	178	179	180
186	189	190	191	193	194	195	196	197	198	199
201	202	203	204	205	206	208	209	214	216	217

220	221	224	225	226	227	228	229	230	231	232
234	235	236	237	238	239	240	241	242	243	244
245	246	247	248	249	250	251	252	253	254	255
256	257	258	259	260	263	264	265	266	267	268
269	270	271	272	273	274	275	276	277	278	279
280	281	282	283	284	285	286	287	288	289	290
291	292	293	294	295	296	297	298	299	301	303

304	305	306	307	309	310	311	313	314	315	316
317	318	319	321	322	323	324	325	326	327	328
329	330	331	332	333	334	338	339	341	342	343
344	345	346	347	348	349	350				

362	433	439	441	444	446	457	460	461	468	474
479	481	488								

510	513	543	550	556	563	564	569	575	580	581
585	589	592	594	597	600	604	608	611		

631	636	651	663	670	671	673	676	681	684	685
687	688	691	692	693	697	699	703			

718	740	743	745	755	757	759	761	763

776	797	799	800

Hence we have the following theorem.

Theorem 2. *There are exactly 300 pairs of complementary coplanar graphs on 7 vertices.*

Authors of [**9**] have reported on 301 such pairs.

4. CASE $n = 8$

The case $n = 8$ is more complex and it could not be easily treated without the aid of a computer.

The table of connected cubic graphs with 12 vertices from [**3**] shows 14 planar 3-connected graphs (identification numbers in the table: 33, 63, 42, 31, 39, 30, 60, 66, 38, 29, 51, 53, 46, 70). The complements of their corresponding triangulations on 8 vertices are presented in Fig. 4 under the names A, B, C, D, E, F, G, H, I, J, K, L, M, N. We shall say that graphs A, B,..., N form the *layer*-0 of the set of lower coplanar graphs on 8 vertices. The graphs are ordered lexicographically according to their eigenvalues. Labelling of vertices has no special meaning; our only concern was that every graph contains edges 1 2 and 1 3.

We then enter a four step recursive extension procedure, starting with the graphs from *layer*-0. To each of 14 graphs of Fig. 4 we add an edge in all possible planar ways, removing graphs that are isomorphic to the already obtained ones.

The mutually different graphs with 11 edges, that are obtained as a result of this first step of extensions, form the *layer*-1. Starting then from these graphs

Fig. 4.

with 11 edges, we repeat the same procedure to find their planar extensions with 12 edges, which form the *layer*-2; and then we obtain in the same way, those with 13 and 14 edges. When a total of 4 layers is added to the *layer*-0 in that way, the graphs in these 5 layers constitute the set of lower coplanar graphs on 8 vertices.

In such a way, more than 100 extended planar successors of the 14 initial *layer*-0 graphs were obtained, but after eliminating isomorphic cases that number was reduced to a total of 67 graphs in *layer*-1. Repeating the process, by taking these 67 graphs as the initial set, we got results represented in the following table:

layer	number of graphs
0	14
1	67
2	242
3	472
4	608

The numbers of graphs in layers 0, 1, 2, 3, 4, according to [**9**], read 14, 61, 234, 489, 602 respectively.

In the sequel we list those graphs by layers, using the following notation. A graph is represented by a sequence of at most five characters. The first character denotes a graph in the *layer*-0 (see Fig. 4), the rest of the characters refer to the edges that have been added to that zero-layer graph. Edges that have been added to the graphs in Fig. 4 are coded as follows:

A 1 4	E 1 8	I 2 6	M 3 5	Q 4 5	U 5 6	Y 6 8
B 1 5	F 2 3	J 2 7	N 3 6	R 4 6	V 5 7	Z 7 8
C 1 6	G 2 4	K 2 8	O 3 7	S 4 7	W 5 8	
D 1 7	H 2 5	L 3 4	P 3 8	T 4 8	X 6 7	

The set of lower coplanar graphs on 8 vertices contains the following graphs,

I in *layer*-0:

A B C D E F G H I J K L M N

I in *layer*-1:

AC	AF	CC	DC	DM	CH	BD	CR	DD	CO
CD	AS	AP	EL	CX	HG	DL	BX	CS	GQ
DG	FH	EQ	ET	FQ	EX	CK	CY	EE	DS
CT	CW	LL	JG	KQ	DE	LC	DK	IL	DY

IC	DW	IR	LJ	EV	KC	FP	MC	IW	GS
FY	JM	KU	HW	FK	IK	IE	NF	IP	LE
JQ	IX	LP	MM	MR	NQ	NP			

I in *layer*-2:

ACM	AFR	CCM	CCH	ACD	DMD	CDC	ACK
BDJ	ACT	CCX	HGL	AFE	AFO	FHM	COM
ASF	COD	CCV	CHO	HGC	EEC	CCE	COX
CYC	CCK	CRO	DMG	BDK	GQE	CXD	CCT
DDL	DGL	CSM	LLG	DEC	ELT	CDS	DYC
BXJ	DCP	EQF	DSD	BDX	CDV	CHK	DEM
FQD	DYM	DCW	DKM	DLQ	ETF	FHJ	CTM
DCZ	ILF	DKH	DCT	APT	CRE	LLN	BXK
KQO	CKR	DGQ	CTH	BXE	FQX	CRT	DSL
FQI	GQU	LCD	LCL	JMF	FPM	DED	CWR
DDP	ETX	IEB	EQK	LJL	CDE	FPC	ASE
CDP	DDK	DWD	APV	CYO	AST	FQJ	EVF
ASK	IKB	ETD	DEX	FKM	CYQ	IRF	CDY
ELW	APX	CWO	ICL	HWB	JMD	KQU	ETB
EED	GQJ	CDW	JGK	EXB	MCF	CTP	LLX
BXY	MCE	LJX	LCH	JGV	KCB	KCQ	CSP
JGQ	CTK	CWP	CXZ	BXW	IEG	EXP	DYL
GSD	ILP	DGK	CKW	MCG	DEG	FYD	LJH
CTS	JGE	IWK	DYR	KCU	DYG	JMG	LEC
CWY	MCR	NFN	DKR	LLV	DGW	DWE	LPL
ICV	FYX	LLE	LJM	JMC	MMH	FQP	LLU
ILK	JQL	IRV	DWY	IXC	IKR	IXG	LCV
JGY	FPE	KUX	LPG	IER	IPN	KQJ	EVE
IWV	FKG	IXL	LPC	DSY	JMU	LPE	DSW
NPB	IRP	KQI	IRD	JGW	NQG	LJE	FPY
IXJ	JMP	MMD	LPJ	NQF	HWT	LCZ	HWK
DYT	FKP	IXD	MCZ	EVZ	KCJ	LJT	IKV
FYW	JME	FPT	IEV	NQE	MML	LJY	LPQ
LEY	LPT	JMW	IKD	LEX	NPV	IPJ	JMT
LPX	MMY	NPG	NPD	IKX	IEX	MRM	NQN
NPC	MRV						

I in *layer*-3:

DMDC	CCMG	ACMJ	HGLM	COMC	CCXM	CCHO
AFRV	CCVH	ACDV	HGCM	CCVR	CCER	CCVG
DMDQ	CDCQ	CCMY	COMX	BDKE	CCKM	CHKM
CCMW	FHMX	BDJP	BDJR	APTM	CCMT	ACTW
BDJV	ACTY	LLNG	CHOR	CCHT	AFEW	ACDE
DSLM	LCLN	APTC	DLQH	CDEC	BXJN	DDPM

EECU	APTR	DEMD	DGLD	AFOX	CCEQ	DCPX
ACDT	CHOS	CDCY	CYCP	FPMC	ACKX	APXM
DYMX	ACKJ	EECQ	CCKY	BDJU	HGLT	FHJX
ELWC	DEMX	IRFB	KQUR	DKMX	HWBC	ELTQ
DSDQ	GQEU	CHKR	DECY	CCXW	AFEJ	CCXT
AFOP	JMGF	LLXG	APTW	DDPX	DGQD	CDPO
CCEW	HGLV	APVF	CDEO	JMFV	DEMP	FQXH
DKHX	EQKU	AFEX	EECW	AFOT	CCEV	BXKN
COMW	DCWR	CYCT	ELTU	ASFT	DKMP	CDEH
CCVY	ASFY	FHMZ	CCVP	CCKW	CODY	COMZ
BXJE	LCLX	BDKX	FHMT	HGCK	ETFU	COXP
KQOW	ETFX	CODW	FQXG	ELTM	CHKY	DYCW
MCEY	DYMW	CCTS	ILKG	JMFU	HGCV	BXEQ
CYOX	DKHP	GQJV	LJMD	DCPW	CKRP	CHOZ
EXBQ	DKME	LLGV	COXZ	DMGW	CYCZ	CHOT
DEMY	DMGK	MCED	EXPF	JGKL	LLEG	BDKU
EEDF	JMFP	DKHY	MCFH	DYGM	LCHU	EEDX
KQOC	CDPX	CSPO	JGKX	JGKT	CDYX	LCDV
EQFV	COXT	COXW	ETXQ	CXDZ	IKRB	DMGZ
DYCS	CWYH	CROW	CTHV	DGLE	EQKW	CTKH
CDEP	DCPZ	BXJY	DEMS	LECD	DECT	NFNJ
DGLT	CXDT	EXPT	CDSP	CXDW	ELWU	GQUD
LLVM	NQGB	EVFB	DEDZ	EQKX	MCEU	DGLY
CDEV	DDLY	DCTY	CTMS	ETDQ	IEBY	DEDW
MCFU	DGLW	ILFX	CDPQ	DDKP	LCLE	ILPR
CWYG	ASTV	DCTS	FYXI	BXJW	LLNV	CDSY
CTPY	JMFE	BXWP	MMHU	ELWP	DYLX	CREW
ELWE	JGVM	EQKD	IKBV	ASKD	LCLV	DEMT
DWDQ	DSWD	ASKV	MCET	FHJW	FPED	CDYE
CKRW	CDYQ	CDVW	APVJ	ETFV	CWPY	DEXY
FQDT	DCWT	EXBT	DLQT	DYMT	ICVD	JMFY
DSDT	CYOE	FQDY	LPGJ	CDPY	ASEY	LJMV
HWKM	DLQY	ICLW	DSLP	KCJR	CDEZ	DWDZ
EQKV	LCDY	BXEU	APTV	KUXP	LCLY	JGQK
ASTO	DKHT	JMUC	JQLX	KQJR	JMDU	ASEX
EQKZ	CYQX	IXCV	KQOK	CDEW	DWDY	KCQF
EXBE	LLET	ASTX	LCZG	CTMZ	JMTF	DSLE
FPCT	DWEY	JGKB	CREZ	LLUJ	LLEJ	IPNG
CDPZ	DWEX	FQIW	IRFY	APXT	FKGQ	DGKP
ETXM	CYQV	FQXT	LECM	LJHU	ICLY	FPCZ
CDPW	HWBV	IXGV	LCLZ	IEBX	DYLS	APTZ
DSLW	JGKM	GQJR	IERY	MCRQ	JMDT	MCEV
HWBD	JMPX	HWKL	ETDV	IKRE	CTPX	MCFZ
JMGU	FPEK	CYOW	KCQH	BXYZ	DGKY	CDYT
JMPK	JMDW	IWVN	JGVP	CKWX	NQEK	EVEU

FKMZ	CXZY	IEVF	LPEH	GSDU	DKRY	LPLX
JGVW	CWYX	MMHY	MCZY	KCJP	DWEZ	FPEQ
LCHP	NPBD	HWBX	FQJP	ICVW	APVZ	FKMT
ASTZ	HWKT	EVZT	DEGW	JMGP	FYXZ	CWPV
IXGW	DGWP	ILPZ	FPEG	FYXW	MCRP	JMEV
FQJT	DGKT	DWYZ	LPJH	FPYW	KCJQ	DEXT
LJEP	EXPV	CYQZ	LECZ	EVZU	MMLD	NPGQ
IEVD	FPYQ	JGYQ	EVEP	MMYU	FQPT	IXLW
NQNU	IKDL	MCRZ	IPJV	DEGZ	ICVY	KQIX
IRVK	LPCT	IEGZ	IKDW	FKPX	CTSZ	JMPW
KCUK	IKDV	IKRS	FYXT	KQIZ	FPTY	JGYU
LPJV	FYXK	LJHY	MCZT	DYGZ	IERD	LPGZ
DGWT	JMPT	LLUY	IPJC	LPQV	FKGY	MRME
IRDK	ICVZ	IERJ	MCZM	DYTQ	LEYQ	JGYW
LPEX	JMTE	IXJW	DSYZ	IEVX	DSWZ	LEXT
IERZ	LEXY	KCJH	LJTU	JMUT	LJYU	LJYT
MMYQ	HWKX	MMYV				

I in *layer*-4:

COMCD	AFEWM	CCMGX	ACTWM	CCVHR	ACDER
DMDCP	ACDVJ	DMDCW	CHKMQ	APXMC	DGLDM
ACMJW	CCMYQ	CCMWQ	DMDQS	DMDCT	ACKJR
ACMJT	AFRVX	JMGFL	HGLVM	CHORX	COMXQ
CDECX	LLNGX	CCERP	CDCYR	BDKEQ	CCKMP
CHKMW	EECQL	ETFXL	COMCW	DSDQX	FPMCX
CCKMX	CCHOP	COMCT	CCHOY	GQEUC	KQURX
DLQHS	CCMWX	BDKEW	AFRVW	DKMEC	AFRVT
ILKGB	APVFR	CCMTX	ACDEY	BDJPX	CCHTX
GQEUP	CCTSG	CCHTO	ACDVW	CDCYE	DGQDX
CCVRY	JMGFV	KQOCR	CDECV	ELWCU	HGCMY
CCVHT	CCVRP	BDKEX	FPMCE	CCEQY	ACDET
ELTUK	CHKMT	APXMW	ILKGR	CCKWR	DMDQW
CCHTY	CCVRT	CCVGY	GQEUY	BDJRY	CDCQY
FHMXW	CDCYS	CCERZ	EECUJ	CYCPO	CCMTS
APXMV	BDJUV	DDPMY	DMDQZ	APTMV	JMPKF
LCLNP	DEMDS	DEMPX	BXJNQ	LPGJL	DMDQY
IRFBV	LCLEN	LLNGV	ELTUP	CODYS	COMXZ
COXPR	DCWRQ	DKMPX	CCKWY	EECQJ	EQKDC
MCFHE	CDEHV	LCLNY	LCDVL	KQOWU	LLEGN
ACKJY	JMFPL	FHMZX	CCVGZ	ACTYX	ETXQU
MCEDX	BDJRW	ELTQW	GQEUN	DYMWD	COMXT
DCPWY	MCEDH	ELTMU	APTMX	DWEXM	LLGVM
JMFPX	CHKRY	DEMXY	BXJYR	BXJWD	DEDZX
FHMTX	APTCD	CYCTQ	DKMXQ	EQKUX	DKMXW

LECDX	DDPXZ	LJMDX	ELTUX	COMWP	MCFHD
DKMEW	CCMTZ	CCEVY	DDPXY	AFOPW	DEMPZ
BXEQR	ACTWZ	CDCYW	DDPXR	APTCX	DKHYX
CCXWY	ACTYZ	CDPOZ	LCDVQ	DKHXW	CCEWO
JMDUV	CTHVQ	DEMXK	LCLNZ	CDEOZ	JGKXT
DSLMZ	CDEOQ	DEMPQ	CHOZR	DMGZX	DCWRY
DLQHY	CDEVX	MCFHY	AFEWZ	ELTUW	CCVPY
AFEJX	ETFUX	DDKPX	JMFVW	CWYHR	LLETP
LLEGP	JMUCD	ELTQZ	ILPRW	CXDZV	CCKYS
BXJEQ	HWBCV	APVFX	ETDQU	DLQHZ	AFOXY
CKRPO	ELWUK	APTCZ	BXJNW	EQFVX	FHMZW
LCHUX	CROWQ	ELTUO	JMGFT	CCEQZ	HWBCX
ETDQX	CDEOY	DKMPQ	ASKDV	FQDTX	EXBTQ
HGLTV	DGLDY	JMFPU	FPEDL	CDEHW	CDPOT
DDLYS	JMFEV	CCEVW	CHOSY	APTRZ	EQKDU
ILKGY	GSDUV	BDJUY	CDEPY	LCHUE	FHJXZ
AFOXZ	HWBCS	FHJXW	KCJPE	CHKYV	JMDWV
DEDWZ	JMGFW	DECYS	APXMZ	EXBTU	ETXQZ
EXBQK	ASFYX	DYMTX	BXKNU	ACKJZ	CCEVT
FPEDZ	DEMPT	EQKDX	CDEZK	FPCZX	GQUDN
CDEOW	DKMEG	ETXQV	LECDY	KQOKR	DYCWZ
DSLPX	NQGBV	CDEVR	CDYEX	EQFVT	EVEUQ
EECWO	DSWDQ	CCXTW	CYOXZ	CREWY	DEMTX
EXBET	CYOER	ELWUP	EXBEQ	DGLYX	CHKRZ
CDEPZ	IRVKG	DCPWZ	ELWUX	DYCSP	HGLVY
ASTOV	JMGUX	BXJWP	DSDQT	LJMDZ	JGKBL
CDPYX	DEDWY	FHJXT	CCEVZ	AFOPZ	CWYHX
COMZY	AFEJZ	DGQDP	CDYQX	CSPOT	KCQFU
DGKYM	IRFYW	JMFUY	DECTW	IPNGW	CCKWS
LCLYX	DWYZX	JGQKX	BXKNV	ELTMZ	CDEWP
BXJWE	MCRPF	CHKYZ	FPCTX	JMTFP	IXCVG
APVFZ	ETFUZ	DLQYX	ETFVX	ASKVX	LLGVP
CXDTQ	ASTVW	APTWX	DWEXY	JGKBQ	CSPOY
DSLWX	BXJEU	MCFHZ	JMFEU	CODYT	CCKWZ
BDKUX	FHMTY	JGKXB	CDYXZ	MCEUZ	ETDQZ
FQXHT	DKHTX	KQOKF	DCWTY	FQXHZ	JMPXL
LJHYG	IEVXB	CWYHT	AFOTZ	CHOZY	IEVFN
FPEDQ	DYCWS	CKRPZ	EQKZW	EQKXP	MMHUX
COXTY	EQFVP	CXZYR	GQUDZ	ELWPO	CYOXW
LLEJX	EVFBT	CDEWX	MCFUV	CDEVZ	DEMSY
JGKMT	DCWTQ	DEDWS	LLENK	EQKDW	ELWEX
KQOKU	LLETZ	GQJVY	CDYES	CTPXE	CKRPV
DMGKT	CDSPZ	LLENY	NPGQE	MCRZX	ASTOX
FPCZQ	DEXYR	IKBVX	CYQXV	LLEJH	LLGVZ
FQDTE	DWEXZ	CDYEZ	DEMSZ	FPEDT	NFNJQ

CKRWX	JGQKP	DYGMZ	DYCSZ	FHJWP	DECTZ
DEGWL	CHOTZ	CDEWY	JGQKW	CDSPY	CREWX
JMGPV	KCQFW	LLETY	FYXIW	MCZYP	CDYTP
HWKLY	LPGJZ	LPGJV	CKWXY	DSWDZ	KCQFX
CDPQY	CTKHZ	DDLYZ	CWYHZ	CDEZT	ETXMZ
IKBVY	LECMT	CDPQW	DYLXZ	MMHUZ	HWKMV
DGKYX	LJMVZ	JMTFE	CDPWY	DWDYL	FPCZW
DGLEZ	MCFZU	ASTVY	DYLXW	JMEVL	ICVDW
DSLPT	ASEXT	MCRPX	EXBTM	DEDWT	IEVDG
EQKVZ	CDEWZ	EXBEW	LLENZ	COXTW	JQLXK
JMPKB	FYXWZ	CDSPW	KCJQE	DWDQY	FQIWY
DSLPZ	ILPZR	JGYWX	ASEYX	JMGPC	LECZJ
LLETV	JQLXY	CDVWZ	ASTVZ	FPYWZ	FKGQT
DYGZX	FPEQX	APVJY	JMPTE	JGKMW	FQIWP
FQDYZ	ASTXY	CDYQW	CXDWT	DGWPY	APTVX
IPJVX	JGVWU	ICLWX	KCQFZ	JMGUK	CYOWS
GSDUZ	ASTOY	JMDUW	ICVDZ	DEGWX	IPNGX
DCWTZ	CTPYV	DCTSZ	JMUCW	DSLEY	FQPTY
DSDTW	MCFZV	KCJHR	IEVFX	IEVDX	JGVPW
MRMED	CREZW	LPJHX	DSLWE	CDYQZ	MMLDR
IRVKY	IKRSY	KUXPH	LPJVX	IEGZP	FKGQY
ICVZL	EVZUW	LPJVQ	ASTOZ	KQIXZ	JMDTU
LPEXM	JGVPY	CTPXW	DLQYZ	APVZT	JMGPU
LJEPX	LPCTX	CWYXZ	FKPXZ	HWKTV	ICVWZ
JMGPY	DWDYT	DGKTW	ASTZY	DSYZP	IKRES
LEXTQ	DYLSW	IERDK	IEVFZ	HWKLV	MCZMG
LEXYJ	FPEGZ	LEXTY	DGWPZ	JMGUY	IRDKV
JMUCT	DEGZW	MCZTU	MMYUZ	DWYZR	FQJPZ
DYTQW	CYQVZ	LPGZX	FPYQJ	JGYWP	LPEHX
IEVDY	MMYQT	FQJTY	DWYZS	MMLDY	JGYUW
IKDWX	IKDVY	HWKXJ	MCZMT	JMUTE	LJTUY
MMYQZ	HWKXO				

We summarize results from this section in the form of the following theorem.

Theorem 3. *There are exactly 1104 pairs of complementary coplanar graphs on 8 vertices.*

5. COMPUTER SEARCH

From the point of view of using system "Graph", the above procedure can be realized through the following algorithm. Starting from the initial graphs, the system adds to them an edge in all possible ways, and automatically removes all graphs which either have multiple edges or are not planar. Detecting isomorphic pairs is also not a problem since "Graph" has corresponding facilities. But in a larger set, especially from the second layer on, it is much better to use graph spectra in finding isomorphic graphs. Generated graphs in one layer are first ordered lexicographically by their spectra. It is thereafter easy to detect cospectral sets of graphs and look inside such sets for still nonisomorphic pairs.

Within the system "Graph" we have a catalogue of cubic graphs with up to 14 vertices from [**3**]. System "Graph" is able to perform several operations and transformations on graphs that have readily produced *layer*-0 of the coplanar graphs on 8 vertices. All additional four layers were generated by using existing commands of the system "Graph". The whole computer search was performed without a need for additional programming efforts, and input of data. We had only to prepare a few command files in a special language of the system "Graph".

Another feasible computer-search approach for the case $n = 8$ would have been to use a computerized catalogue of all graphs on 8 vertices, and check each graph on being coplanar. According to [**8**] the number of graphs on 8 vertices is 12344. However it is sufficient to check graphs with 10 (663 graphs), 11 (980 graphs), 12 (1312 graphs), 13 (1557 graphs) and 14 edges (1646 graphs); i.e. a total of 6158 graphs. We had a catalogue of graphs on 8 vertices, but irrespective of that we decided to use the approach described in this paper.

6. CONCLUSION

We summarize our results in the form of the following theorem.

Theorem 4. *There are exactly 2976 coplanar graphs.*

Proof. We use data from [**8**] on the numbers of graphs with a given number of vertices, Proposition 1, and Theorems 1 and 2.

In the comparative table below g_n is the number of graphs on n vertices as reproduced from [**8**], and p_n is the number of coplanar graphs on n vertices.

n	1	2	3	4	5	6	7	8
g_n	1	2	4	11	34	156	1044	12344
p_n	1	2	4	11	32	128	600	2198
c_n	1	1	2	6	17	64	300	1104

The figures for p_n are obvious for $n = 1, 2, \dots, 5$. For $n = 6, 7, 8$ the number of coplanar graphs follows from Sections 2, 3, and 4 respectively.

This completes the proof.

The number c_n of complementary pairs of planar graphs on n vertices is also included in the above table. These numbers include pairs consisting of selfcomplementary graphs. All selfcomplementary graphs up to 8 vertices are planar (checked by "Graph"). There are 1, 1, 2, 10 selfcomplementary graphs on 1, 4, 5, 8 vertices respectively.

Our approach to coplanar graph can be used to prove that the thickness of K_9 is equal to 3. There are 50 cubic planar 3-connected graphs on 14 vertices [**3**] giving rise to 50 triangulations on 9 vertices. We checked that complements of all these triangulations are nonplanar, thus showing that there are no coplanar graphs on 9 vertices (c.f. note added in proof of [**2**] where it was noted that W. T. Tutte has used triangulations on 9 vertices to prove that the thickness of K_9 is equal to 3).

REFERENCES

1. J. Akiyama, F. Harary: *Graph and its complement with specified properties VII: a survey, The theory and applications of graphs.* Proc. Fourth. Internat. Conf. on Theory and Appl. of Graphs, ed. G. Chartrand, Y. Alavi, D. L. Goldsmith, L. Lesiak-Foster, D. R. Lick, John Wiley & Sons, New York—Chichester—Brisbane—Toronto—Singapore, 1981, 1–12.
2. J. Battle, F. Harary, Y. Kodama: *Every planar graph with nine points has a nonplanar complement.* Bull. Amer. Math. Soc., **68** (1962), 2, 569–571.
3. F. C. Bussemaker, D. Cvetković, J. J. Seidel: *Computer investigation of cubic graphs.* Technological University Eindhoven, T. H. –Report **76–WSK–01**.
4. D. Cvetković: *A survey of recent research in graph theory supported by the expert system "Graph".* Contributed talk at the Tenth British Combinatorial Conference 22–26 July 1985, University of Glasgow, Glasgow.
5. D. Cvetković, M. Doob, I. Gutman, A. Torgašev: *Recent results in the theory of graph spectra.* North-Holland, Amsterdam, 1988.
6. D. Cvetković, L. Kraus, S. Simić: *Discussing graph theory with a computer I, Implementation of graph theoretic algorithms.* Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. Fiz., **716–734** (1981), 100–104.
7. D. Cvetković, M. Petrić: *A table of connected graphs on six vertices.* Discrete Math., **50** (1984), 1, 37–49.
8. F. Harary: *Graph theory.* Adison-Wesley, Reading, MA, 1969.
9. D. Y. Li, J. Z. Mao, B. L. Liu: *On the Akiyama-Harary problem about complementary planar graphs.* J. Central China Normal Univ. Natur. Sci., **22** (1988), 1, 1–8.
10. Z. Radosavljević: *Coplanar graphs.* Vertrag an XI österreichischem Mathematikerkongress, Graz, 16–20 September 1985.
11. Z. Radosavljević: *Contribution to the theory and practice of the usage of an expert system in investigations in the theory of graphs.* (Serbo-Croatian), Doctoral Thesis, University of Belgrade, Belgrade, 1987.

12. S. Simić, Z. Radosavljević: *Some computer-aided investigations in graph theory.* Proc. Symposium on Optimization, Design of Experiments and Graph Theory, Indian Institute of Technology, Bombay, Dec. 15–17, 1986, 307–313.

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ADDED IN PROOF

Yang Yuansheng (Dalian University of Technology and University of Stirling) has recently obtained the same computational results comerning generation of coplanar graphs using computer programs developed by himself.