Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 2 (1991), 42–48.

CORRECTIONS AND SUPPLEMENTS OF SOME DETAILS IN TWO FORMER PAPERS

Dušan D. Adamović

This note contains corrections of some minor mistakes in papers [1] and [2] and also supplements and improvements of corresponding points in these papers.

0. In this note we shall give, as suggested by the title, not only corrections of noticed mistakes but also supplements and improvements of corresponding points in previously published papers [**1**] and [**2**]. Those mistakes, observed by the author immediately after the publications of [**1**] and [**2**], have not seriously damaged the main results of these papers, so that the corresponding places, after necessary corrections, can be replaced by similar propositions, as will be seen in what follows.

The present author is reputed to be a rather strict, even pedantic reviewer and critic of mathematical texts, and so it was completely natural and appropriate that he should apply his usual standards to his own papers.

1. Theorems 1 and 2 in [**1**] determine general solutions of functional equations

$$f\big(f(x+y)\big) = f(x) + f(y) \tag{1}$$

and

$$f(x+y) = f\big(f(x)\big) + f\big(f(y)\big) \tag{2}$$

respectively, where, in both cases, the unknown function f can be *real* (i.e. $f\colon \mathsf{R} \to \mathsf{R}$, R desinating the set of all real numbers), or *complex* ($f\colon \mathsf{C} \to \mathsf{C}$, C desinating the set of all complex numbers). Let us call the first possibility *real case* and the second one *complex case*. By using the system of functional equations

$$\begin{cases} g(x+y) = g(x) + g(y) \\ g\big(g(x)\big) = g(x) \end{cases} \qquad (g\colon \mathsf{R} \to \mathsf{R}, \quad \text{or} \quad g\colon \mathsf{C} \to \mathsf{C}), \tag{3}$$

1991 Mathematics Subject Classification: 39B12, 40A05

one can, on account of Lemma 1 in [**1**], state the mentioned theorems in the following manner, different from that in which they really have been stated in [**1**]. (The formulations of Theorems 1 and 2 given in [**1**] are directly, without regard to Lemma 1, more informative than the formulations we give now, but these last formulations are more suitable for the following exposition; otherwise, on the basis of Lemma 1 in [**1**] the first formulations can be deduced from the second ones, and vice versa).

I. *The general solution of the functional Equation (1) is given by* $f(x) = g(x)+\lambda$ *(*g *arbitrary solution of (3), constant* λ *arbitrary element of* $g(\mathsf{R})$*, respectively of* $g(\mathsf{C})$*).*

II. *The general solution of the functional Equation (2) is given by* $f(x) = g(x)$ *(*g *arbitrary solution of (3)).*

In [**1**], under the titles *"Corollary of Theorem 1"* and *"Corollary of Theorem 2"* respectively, (without proof) the following sentences were formulated:

I′. *All continuous solutions of Equation (1) are given by* $f(x) = x+\lambda$ *(*λ *arbitrary real, or complex constant),* $f(x) = 0$*.*

II″. *Equation (2) gas only two continuous solutions, given by* $f(x) = x$ *and by* $f(x) = 0$*.*

However, *statements* I' *and* II'' *are not true in the complex case, while they remain true in the real case.*

This incorrectness resulted from the circumstance that, considering the question of continuous solutions of (1) and (2), we presumed by mistake that even in the complex case ($f\colon \mathsf{C} \to \mathsf{C}$) the general continuous solution of Cauchy's functional equation

$$g(x+y) = g(x) + g(y) \tag{4}$$

is given by

$$f(x) = Cx \qquad (C \text{ arbitrary complex constant}).$$

In fact, *the general continuous solution of (4) in the complex case can be expressed in the following manner*

$$g(x) = (\alpha + i\gamma)\,\Re x + (\beta + i\delta)\,\Im x \qquad (\alpha,\ \beta,\ \gamma,\ \delta \textit{ arbitrary real constants}). \tag{5}$$

Using this fact and statements I and II, one arrives at the following correct version of corollaries of theorems in [**1**].

Corollary of Theorem 1. *All continuous solutions of Equation (1) in the complex case are given by:*

$$f(x) = x + \lambda \qquad (\lambda \textit{ arbitrary complex constant});$$

$$f(x) = 0;$$

$$f(x) = \Re x + \lambda \qquad (\lambda \textit{ arbitrary real constant});$$

$$f(x) = i\,\Im x + i\mu \qquad (\mu \textit{ arbitrary real constant});$$

$$f(x) = \big(1 + i(1-\alpha)/\beta\big)\big(\alpha\,\Re x + \beta\,\Im x\big) + \lambda$$

$\big($*α and $\beta \neq 0$ arbitrary real constants, constant λ arbitrary element of* $\big\{\big(1 + i(1-\alpha)/\beta\big)\ t\colon t \in \mathrm{R}\big\}\big)$;

$$f(x) = \big((1-\beta)/\alpha + i\big)\big(\alpha\,\Re x + \beta\,\Im x\big) + \lambda$$

$\big($*$\alpha \neq 0$ and β arbitrary real constants, constant λ arbitrary element of* $\big\{\big((1-\beta)/\alpha + i\big)\ t\colon t \in \mathrm{R}\big\}\big)$.

In the real case, all continuous solutions of (1) are given by:

$$f(x) = x + \lambda \qquad (\lambda \text{ arbitrary real constant});$$

$$f(x) = 0.$$

Corollary of Theorem 2. *All continuous solutions of Equation (2) are given by:*

$$f(x) = x; \qquad f(x) = 0; \qquad f(x) = \Re x; \qquad f(x) = i\,\Im x;$$

$$f(x) = \big(1 + i(1-\alpha)/\beta\big)\big(\alpha\,\Re x + \beta\,\Im x\big) \qquad (\alpha \text{ and } \beta \neq 0 \text{ arbitrary real constants});$$

$$f(x) = \big((1-\beta)/\alpha + i\big)\big(\alpha\,\Re x + \beta\,\Im x\big) \qquad (\alpha \neq 0 \text{ and } \beta \text{ arbitrary real constants}).$$

In the real case all continuous solutions of (2) are given by

$$f(x) = x; \qquad f(x) = 0.$$

Proof. We shall restrict ourselves to the proof of the assertions concerning the complex case, because those concerning the real case can simply be verifed.

In view of Theorems I and II, in order to find all continuous solutions of (1) resp. (2) it is sufficient to find all continuous solutions $g(x)$ of the system (3) of functional equations and include them into the formulas for the general solution given in I and II. On the other hand, all continuous solutions of (3) can be obtained if one determines all values of real constants in (5) for which the second equation in (3)

$$g\big(g(x)\big) = g(x)$$

it satisfied. It is easy to establish that, with designations $\xi = \Re x$, $\eta = \Im x$, the last is equivalent to

$$\alpha(\alpha\xi + \beta\eta) + \beta(\gamma\xi + \delta\eta) = \alpha\xi + \beta\eta, \qquad (\xi, \eta \in \mathrm{R}),$$

$$\gamma(\alpha\xi + \beta\eta) + \delta(\gamma\xi + \delta\eta) = \gamma\xi + \delta\eta, \qquad (\xi, \eta \in \mathrm{R}).$$

It is clear that this condition is equivalent to the following system of equations for real constants:

$$(6) \qquad \alpha^2 + \beta\gamma = \alpha, \qquad \alpha\beta + \beta\delta = \beta, \qquad \gamma\alpha + \delta\gamma = \gamma, \qquad \gamma\beta + \delta^2 = \delta.$$

Under the assumption $\beta \neq 0$, the first Equation (6) becomes

$$\gamma = \frac{\alpha - \alpha^2}{\beta}, \tag{7}$$

and the second one

$$\alpha + \delta = 1; \tag{8}$$

it follows from (8) $\alpha^2 - \delta^2 = (\alpha - \delta)(\alpha + \delta) = \alpha - \delta$, that is $\alpha^2 - \alpha = \delta^2 - \delta$, which means that the fourth Equation (6) is satisfied too; (8) also implies that the third equation is satisfied. So, under the assumption $\beta \neq 0$, (6) is equivalent to

$$\delta = 1 - \alpha, \qquad \gamma = \frac{\alpha - \alpha^2}{\beta} \quad \left(= (1-\alpha)\frac{\alpha}{\beta}\right).$$

Similarly, under the assumption $\gamma \neq 0$, (6) is equivalent to

$$\alpha = 1 - \delta, \qquad \beta = \frac{\delta - \delta^2}{\gamma} \quad \left(= (1-\delta)\frac{\delta}{\gamma}\right).$$

Further, if $\beta = \gamma = 0$, the second and the third Equation (6) are satisfied, and the first and the fourth equation become $\alpha^2 = \alpha$ and $\delta^2 = \delta$, and are satisfied if and only if $\alpha = \delta = 1$, or $\alpha = \delta = 0$, or $\alpha = 1$ and $\delta = 0$, or $\alpha = 0$ and $\delta = 1$.

It follows from all previously said and established that both propositions are true.

2. One of results in [**2**] was formulated as follows:

"**Proposition 2.** *For each complex number z and all natural numbers m,*

$$\sum_{k=0}^{n} \frac{z^k}{k^m}\binom{n}{k} \sim \begin{cases} \dfrac{(z+1)^{m+n}}{(zn)^m} \ (n \to \infty), & \text{if } |z+1| > 1, \\ -\dfrac{\log^m n}{m!} \ (n \to \infty), & \text{if } |z+1| \leq 1 \wedge z \neq 0. \end{cases}$$

If $|z+1| \leq 1 \wedge z \neq 0$, we have, more precisely,

$$\sum_{k=1}^{n} \frac{z^k}{k^m}\binom{n}{k} = -\frac{\log^m n}{m!} - \big[\log(-z) + C\big]\frac{\log^{m-1} n}{(m-1)!} + \lambda(m,z) + O\left(\frac{\log^{m-1} n}{n}\right) \tag{9}$$

($n \to \infty$; z complex number, $m \in \mathbf{N}$), where C denotes Euler's *constant, $\lambda(m,z)$ does not depend on z and the determination of the complex logarithm is such that* $\log 1 = 0$."

The part of this proposition stated by the second sentence (i.e. by the text we have designated here by $\big[$ *is not true.*

More precisely, even this statement would be true if the sum of two last members in the right side of (9) were replaced by $O\big(\log^{m-2} n\big)$ and if everything concerning $\lambda(m,z)$ were omitted from the further text.

In fact, what we first noticed was the incorrectness of the part of the proof corresponding to this part of the proposition, and also the fact that this part of the proposition, changed as we said, can be proved by slight modifications of the proof. Afterwards, S. Simić in [**3**], determining the infinite asymptotic exposition of the expression on the left side under the condition $|z+1| \le 1 \wedge z \ne 0$, proved effectively that (9) is not true.

Therefore:

The statement formulated by the second part of Proposition 2 in [**2**] *need be replaced by the following:*

T. *If* $|z+1| \le 1 \wedge z \ne 0$, *we have, more precisely,*

$$\sum_{k=1}^{n} \frac{z^k}{k^m}\binom{n}{k} = -\frac{\log^m n}{m!} - \big[\log(-z) + C\big]\frac{\log^{m-1} n}{(m-1)!} + O\big(\log^{m-2} n\big) \qquad (n \to \infty) \tag{10}$$

(z *complex number*, $m \in \mathrm{N}$), *where* C *denotes* Euler's *constant and the determination of the complex logarithm is such that* $\log 1 = 0$.

Proof of T. In this proof, as in that exposed in [**2**], the following two known results will be used (the second of them is here, as in [**2**], formulated in a somewhat more precise form than necessary for the exposition which follows).

$\mathbf{R_1}$. (Proposition 1 in [**2**]). The equality

$$\sum_{k=1}^{n} \frac{z^k}{k^m}\binom{n}{k} = \sum_{k_1=1}^{n} \frac{1}{k_1} \sum_{k_2=1}^{k_1} \frac{1}{k_2} \cdots \sum_{k_m=1}^{k_{m-1}} \frac{1}{k_m}\big[(z+1)^{k_m} - 1\big]$$

holds for each complex number z and for all natural numbers m and n.

$\mathbf{R_2}$. (Immediate Corollary of Theorem (8.4) in [**4**], p. 32). For $\alpha \ne -1$ and real,

$$\sum_{k=1}^{n} \frac{\log^\alpha k}{k} = \frac{\log^{\alpha+1} n}{\alpha+1} + \varphi(\alpha) + \Delta_n(\alpha) \qquad (n \in \mathrm{N}),$$

where $\varphi(\alpha)$ does not depend on n, the sequence $\big(\Delta_n(\alpha)\big)_{n\in\mathrm{N}}$ is decreasing for n large enough and

$$\Delta_n(\alpha) = O\left(\frac{\log^\alpha n}{n}\right) \qquad (n \to \infty).$$

After these remarks, the further text of our proof will be the text of the corresponding part of the proof in [**2**] with necessary changes.

Let

$$|z+1| \le 1 \ \wedge \ z \ne 0.$$

This condition implies the convergence of the series $\sum_{k=1}^{\infty} \frac{(z+1)^k}{k}$ to the sum $-\log\bigl[1-(z+1)\bigr] = -\log(-z)$, with the determination choosen as above. Hence and in view of R_1 and R_2, we have

$$\begin{aligned}
(11)\qquad \sum_{k=1}^{n} \frac{z^k}{k}\binom{n}{k} &= \sum_{k=1}^{n} \frac{(z+1)^k-1}{k} = -\sum_{k=1}^{n}\frac{1}{k} + \sum_{k=1}^{n}\frac{(z+1)^k}{k} \\
&= -\left[\log n + C + O\left(\frac{1}{n}\right)\right] + \sum_{k=1}^{\infty}\frac{(z+1)^k}{k} + O\left(\frac{1}{n}\right) \\
&= -\log n - \bigl[\log(-z)+C\bigr] + O\left(\frac{1}{n}\right) \qquad (n\to\infty).
\end{aligned}$$

Thus, (10) is true for $m=1$. Supposing the validity of (10) for a fixed value of m, we obtain, using R_1 and R_2,

$$\begin{aligned}
\sum_{k=1}^{n} \frac{z^k}{k^{m+1}}\binom{n}{k} &= \sum_{k=1}^{n}\frac{1}{k}\left\{\sum_{k_1=1}^{k}\frac{1}{k_1}\sum_{k_2=1}^{k_1}\frac{1}{k_2}\cdots\sum_{k_m=1}^{k_{m-1}}\frac{1}{k_m}\bigl[(z+1)^{k_m}-1\bigr]\right\} \\
&= \sum_{k=1}^{n}\frac{1}{k}\sum_{l=1}^{k}\frac{z^l}{l^m}\binom{k}{l} \\
&= \sum_{k=1}^{n}\frac{1}{k}\left\{-\frac{\log^m k}{m!} - \bigl[\log(-z)+C\bigr]\frac{\log^{m-1}k}{(m-1)!} + \alpha_k \log^{m-2} k\right\} \\
&= -\frac{1}{m!}\sum_{k=1}^{n}\frac{\log^m k}{k} - \frac{\log(-z)+C}{(m-1)!}\sum_{k=1}^{n}\frac{\log^{m-1}k}{k} \\
&\quad + O\left(\sum_{k=1}^{n}\frac{\log^{m-2}k}{k}\right) \\
&= -\frac{1}{m!}\left[\frac{\log^{m+1}n}{m+1} + \varphi(m) + O\left(\frac{\log^m n}{n}\right)\right] \\
&\quad - \frac{\log(-z)+C}{(m-1)!}\left[\frac{\log^m n}{m} + \varphi(m-1) + O\left(\frac{\log^{m-1} n}{n}\right)\right] \\
&\quad + O\bigl(\log^{m-1} n\bigr)
\end{aligned}$$

$$= \frac{\log^{m+1} n}{(m+1)!} - \big[\log(-z) + C\big]\frac{\log^m n}{m!} + O\big(\log^{m-1} n\big) \qquad (n \to \infty),$$

where $(\alpha_n)_{n\in\mathrm{N}}$ is a bounded sequence. Especially, for $m = 1$, $\log^{m-2} k$ in the third and the fourth row of these formulae ought to be replaced by k^{-1} (see (11)) and so all which follows will be correct in this case, too.

This completes our inductive proof.

REFERENCES

1. D. D. Adamović: *Résolutions de deux équations fonctionnelles proches de l'équation de Cauchy.* Mat. Vesnik, **9** (**24**) (1972), 1, 19–22.
2. D. D. Adamović: *An Identity and Asymptotic Behaviour of Some Sequences.* Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. Fiz., **544**–**576** (1976), 149–154.
3. Slavko Simić: *Asymptotic Expansion of Some Complex Sequences.* Publ. Inst. Math. (Beograd), **39** (**53**) (1986), 119–128.
4. , 1965.

Faculty of Sciences,
University of Belgrade,
Studentski trg 16,
11000 Belgrade,
Yugoslavia

(Received October 25, 1990)