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A NOTE ON q–GAMMA FUNCTION

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In this paper certain properties of q–gamma function are studied. We prove relation (2.1) which relates the q–gamma functions in the cases $0<q<1$ and $q>1$. Some applications of this relation are also given. First, we derive an analogue of the Bohr–Mollerup theorem for the q–gamma function in the case $q>1$. Our result is more general than the Moak's [3]. Also, the behavior of the q–gamma function with respect to q is considered.

1. INTRODUCTION

F. H. Jackson [**1**] defined a generalization of the factoriel function by

$$(n!)_q = 1(1+q)\cdots(1+q+\cdots+q^{n-1})$$

for $q>0$, as well as a generalization of the gamma function by

$$x \mapsto \Gamma_q(x) = \frac{(q;\, q)_\infty (1-q)^{1-x}}{(q^x;\, q)_\infty}, \qquad (0<q<1) \tag{1.1}$$

$$x \mapsto \Gamma_q(x) = q^{\binom{x}{2}} \cdot \frac{(q^{-1};\, q^{-1})_\infty (q-1)^{1-x}}{(q^{-x};\, q^{-1})_\infty}, \qquad (q>1) \tag{1.2}$$

where $(a;\, q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$, and x is real.

R. Askey [**2**] considered only the case $0<q<1$ and showed that q–gamma function has many similar properties as classical gamma function. For example, an analogue of Bohr–Mollerup theorem is

Theorem A. *Let f be the function which satisfies:*

$$f(x+1) = \frac{1-q^x}{1-q} f(x), \qquad \text{for some} \quad q, \quad 0<q<1, \tag{1.3}$$

$$f(1) = 1, \tag{1.4}$$

$$x \mapsto \log f(x) \qquad \text{is convex for} \quad x>0. \tag{1.5}$$

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Then $f = \Gamma_q$.

D. S. Moak [**3**] studied the case $q > 1$. He gave two analogues of Bohr–Mollerup theorem. He showed that the function f, satisfying (1.3), (1.4) ($q > 1$) and for $x > 0$ $\dfrac{\mathrm{d}^3}{\mathrm{d}x^3}\log f(x) \le 0$ or $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\log f(x) \ge \log q$, is the q–gamma function.

The behavior of q–gamma function, when q changes, was considered by R. Askey [**2**] in the case $0 < q < 1$ and by D. S. Moak [**3**] for $q > 1$.

2. RELATION BETWEEN Γ_p AND Γ_q, $pq = 1$

Lemma. *Let* $p,\ q > 0,\ p,\ q \ne 1,\ pq = 1$. *Then*

$$\Gamma_p(x) = q^{-\binom{x-1}{2}}\Gamma_q(x). \tag{2.1}$$

Proof. For example, let $0 < q < 1$. Then $p > 1$ and we have

$$\begin{aligned}\Gamma_p(x) &= p^{\binom{x}{2}} \cdot \frac{(p^{-1};\, p^{-1})_\infty (p-1)^{1-x}}{(p^{-x};\, p^{-1})_\infty} \\ &= q^{-\binom{x}{2}} \cdot \frac{(q;\, q)_\infty (1-q)^{1-x} q^{x-1}}{(q^x;\, q)_\infty} = q^{-\binom{x-1}{2}}\Gamma_q(x),\end{aligned}$$

which proves the Lemma.

Now, we shall give some applications of the above lemma. First, we shall prove the analogue of Bohr–Mollerup theorem for $q > 1$, which holds under weaker supposition than Moak's Theorem 2 from [**3**].

Theorem 1. *Let* f *be a positive function defined on* $(0, +\infty)$ *which satisfies (1.3), (1.4) for some* $q > 1$, *and*

$$x \mapsto -\frac{\log q}{2}x^2 + \log f(x) \qquad \text{is convex for} \quad x > 0, \quad q > 1 \tag{2.2}$$

Then $f = \Gamma_q$.

Proof. Substituting

$$F(x) = q^{-\binom{x-1}{2}} f(x), \tag{2.3}$$

into (1.3), (1.4), (2.2), we find

$$F(x+1) = \frac{1-p^x}{1-p}F(x), \qquad \left(p = \frac{1}{q},\ 0 < p < 1\right)$$

$$F(1) = 1$$

and that the function $x \mapsto (1-\frac{3}{2}x)\log q + \log F(x)$ is convex for $x > 0$. This implies that function $x \mapsto \log F(x)$ is convex for $x > 0$ and we conclude that the function F satisfies conditions of Theorem A. It follows that $F = \Gamma_p$, $0 < p < 1$, and we obtain, by using (2.3), that $f = \Gamma_q$, which proves the theorem.

Supposing condition (2.2) reduces to $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log f(x) \geq \log q$, and we obtain the result of D. S. MOAK [3].

Similarly, as the Theorem 1, by using MOAK'S Theorem 1 in [**3**], it can be proved that the function f satisfying (1.3), (1.4) and $\frac{\mathrm{d}^3}{\mathrm{d}x^3}\log f(x) \leq 0$, for $x > 0$ is the q–gamma function, $0 < q < 1$.

Also, it is easy to prove by using (2.1), that the LEGENDRE duplication formula, GAUSS multiplication formula for q–gamma function, which are obtained by R. ASKEY [**2**] in the case $0 < q < 1$, hold, in unchanged form, for $q > 1$.

3. THE BEHAVIOR OF Γ_q AS THE FUNCTION OF q

Theorem 2.

(i) If $0 < r < q < 1$ *then*

$$r^{\binom{x-1}{2}}\Gamma(x) \leq \left(\frac{r}{q}\right)^{\binom{x-1}{2}}\Gamma_q(x) \leq \Gamma_r(x) \leq \Gamma_q(x) \leq \Gamma(x), \tag{3.1}$$

for $0 < x \leq 1$ *or* $x \geq 2$*;*

$$\Gamma(x) \leq \Gamma_q(x) \leq \Gamma_r(x) \leq \left(\frac{r}{q}\right)^{\binom{x-1}{2}}\Gamma_q(x) \leq r^{\binom{x-1}{2}}\Gamma(x), \tag{3.2}$$

for $1 \leq x \leq 2$.

(ii) If $r > q > 1$, *then*

$$\Gamma(x) \leq \Gamma_q(x) \leq \Gamma_r(x) \leq \left(\frac{r}{q}\right)^{\binom{x-1}{2}}\Gamma_q(x) \leq r^{\binom{x-1}{2}}\Gamma(x), \tag{3.3}$$

for $0 < x \leq 1$ *or* $x \geq 2$*;*

$$r^{\binom{x-1}{2}}\Gamma(x) \leq \left(\frac{r}{q}\right)^{\binom{x-1}{2}}\Gamma_q(x) \leq \Gamma_r(x) \leq \Gamma_q(x) \leq \Gamma(x), \tag{3.4}$$

for $1 \leq x \leq 2$.

Proof. Some of the above inequalities are proved in [**2**], [**3**].

Let $0 < r < q < 1$. Since $\frac{1}{r} > \frac{1}{q} > 1$ we have (see [**3**])

$$\Gamma_{1/r}(x) \geq \Gamma_{1/q}(x) \geq \Gamma(x),$$

for $0 < x \leq 1$ or $x \geq 2$. Using (2.1) we obtain

$$r^{-\binom{x-1}{2}}\Gamma_r(x) \geq q^{-\binom{x-1}{2}}\Gamma_q(x) \geq \Gamma(x),$$

wherefrom follows

$$\Gamma_r(x) \geq \left(\frac{r}{q}\right)^{\binom{x-1}{2}} \Gamma_q(x) \geq r^{\binom{x-1}{2}}\Gamma(x).$$

Other inequalities in (3.1) are proved in [**2**].

Inequalities (3.2), (3.3), (3.4) can be proved similarly.

A direct consequence of the Theorem 3 is the following

Theorem 4. *Let $q > 0$, $q \neq 1$. Then, for $x > 0$*

$$\Gamma_q(x) = \Theta^{\binom{x-1}{2}}\Gamma(x), \tag{3.5}$$

where Θ is a function of q, x, such that

$$\min(1,\, q) \leq \Theta \leq \max(1,\, q).$$

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