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CHARACTERISTIC AND MATCHING POLYNOMIALS OF SOME BIPARTITE GRAPHS

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We add a fixed number of vertices of degree 1 to each vertex from one part of a bipartite graph. We study characteristic, matching and some related polynomials for graphs obtained in this way.

1. INTRODUCTION

Let G be a bipartite graph with a vertices $v_1, \ldots, v_a$ of one color and b vertices $v_{a+1}, \ldots, v_{a+b}$ of the other color. For the present consideration it is immaterial whether $a \geq b$ or $a \leq b$.

Let R_1 and R_2 be two rooted (not necessarily bipartite) graphs. Then the graph product $G[R_1, R_2]$ is obtained by identifying each of the vertices v_i, $i = 1, \ldots, a$ of G with the root of a copy of R_1 and by identifying each of the vertices v_i, $i = a+1, \ldots, a+b$ of G with the root of a copy of R_2.

The characteristic and the matching polynomials as well as the respective spectra of $G[R_1, R_2]$ were examined in a few previous publications [**2**], [**4**], [**5**]. In particular, it was shown [**4**] that for $\pi = \phi$ or $\pi = \alpha$,

$$\begin{aligned}\pi\left(G\left[R_1,\, R\right]\right) = \left[\frac{\pi(R_1)}{\pi(R_2)}\right]^{\frac{(a-b)}{2}} \left[\pi\left(R_1^*\right)\pi\left(R_2^*\right)\right]^{\frac{(a+b)}{2}} \cdot \\ \cdot\, \pi\left(G,\, \left[\pi(R_1)\pi(R_2)\pi\left(R_1^*\right)^{-1}\pi\left(R_2^*\right)^{-1}\right]^{\frac{1}{2}}\right),\end{aligned} \tag{1}$$

where $\phi(H) = \phi(H, x)$ and $\alpha(H) = \alpha(H, x)$ stand for the characteristic and the matching polynomial of a graph H, respectively, and where R_i^* denotes the graph obtained by deleting the root from R_i, $i = 1, 2$.

If R_1 is the star with $p+1$ vertices rooted at the vertex of degree p, and R_2 is the one–vertex graph, then we denote the graph product $G\left[R_1, R_2\right]$ by $G[p]$. It is

AMS Subject Classification (1980): 05C50 clear that $G[p]$ is obtained by attaching p new vertices of degree one to each vertex v_i, $i = 1, \dots, a$ of G; hence $G[p]$ has $(a+1)p + b$ vertices.

Formula (1) immediately furnishes

$$\pi(G[p]) = x^{pa-(a-b)}(x^2 - p)^{\frac{(a-b)}{2}} \pi(G, \sqrt{x^2 - p}); \qquad \pi = \phi,\ \alpha. \tag{2}$$

In this paper we first point out a hitherto unnoticed property of $\pi(G[p])$, $\pi = \phi,\ \alpha$. Then we demonstrate that the relation (1) remains valid if the symbol π is interpreted as a much more general graph polynomial.

The characteristic and the matching polynomial of the graph G can be written in the form

$$\pi(G) = \sum_{k=0}^{a} (-1)^k c_\pi(G,k) x^{a+b-2k}; \qquad \pi = \phi,\ \alpha \tag{3}$$

where $c_\pi(G,k) \geq 0$. Bearing in mind eq. (2) we also have

$$\pi(G[p]) = x^{pa-(a-b)} \sum_{k=0}^{a} (-1)^k c_\pi(G[p],k) x^{2(a-k)}; \qquad \pi = \phi,\ \alpha. \tag{4}$$

It is not difficult to see that $c_\pi(G[p],k) > 0$ for $0 \leq k \leq a$, $\pi = \phi,\ \alpha$.

The usual way in which the characteristic and the matching polynomials of a graph are defined can be found, for instance, in [**1**]. Here we put forward a less common recursive characterization of these graph polynomials which, however, is more appropriate for the present considerations.

Definition 1. *If H is a graph with n vertices, $n \geq 0$, and no edges then the matching polynomial of H is equal to x^n. If H possesses an edge, say e connecting the vertices u and v, then the matching polynomial of H conforms to the recursion relation*

$$\alpha(H) = \alpha(H-e) - \alpha(H-u-v). \tag{5}$$

Definition 2. *If H is an acyclic graph then its characteristic and matching polynomials coincide. If the graph H possesses circuits then the characteristic polynomial of H conforms to the recursion relation*

$$\phi(H) = \phi(H-e) - \phi(H-u-v) - 2 \sum_{j \in I(e)} \phi(H - C_j). \tag{6}$$

Here and later $C_1, \dots, C_r$ are the circuits of the graph considered and C_j, $j \in I(e)$ are the circuits containing the edge e. Note that $I(e) \subseteq \{1, \dots, r\}$ and that $I(e)$ may be an empty set; in that latter case the summation term on the right–hand side of (6) vanishes.

Eqs. (5) and (6) provide a motivation for a generalization and unification of the concepts of matching and characteristic polynomials. We define a graph polynomial $\mu(H) = \mu(H,x)$ as follows [**3**], [**6**].

Definition 3. *If H is an acyclic graph then its characteristic, matching and μ–polynomials coincide. If the graph H possesses circuits then the μ–polynomial of H conforms to the recursion relation*

$$\mu(H) = \mu(H-e) - \mu(H-u-v) - 2\sum_{j\in I(e)} t_j\mu(H-C_j), \tag{7}$$

where t_j is a parameter associated with the circuit C_j, $j = 1,\ldots,r$. If $I(e) = \emptyset$ then the summation term on the right–hand side of (7) is equal to zero.

Comparing eqs. (5)–(7) it is evident that

$$\mu(H) = \alpha(H), \qquad \text{if} \quad r = 0 \quad \text{or if} \quad t_1 = \cdots = t_r = 0,$$
$$\mu(H) = \phi(H), \qquad \text{if} \quad r = 0 \quad \text{or if} \quad t_1 = \cdots = t_r = 1,$$

and thus $\mu(H)$ is a proper generalization of both the matching and the characteristic polynomials. Further, if H is a bipartite graph ($H = G$), then its μ–polynomial has the form (3), $\pi = \mu$. Note, however, that $c_\mu(G,\,k)$ need not be non–negative – their signs depend on the actual values of the parameters t_i, $i = 1,\ldots,r$.

The basic properties of the μ–polynomial are determined in [**3**], [**6**].

2. A RESULT FOR THE GRAPH POLYNOMIALS OF $G[p]$.

Let the coefficients of the characteristic and the matching polynomials of the graphs G and $G[p]$ be determined by means of (3) and (4), $\pi = \phi$ and $\pi = \alpha$. Then

$$c_\pi(G[p],\,a) = \sum_{m=0}^{a} c_\pi(G,\,m)p^{a-m} \tag{8}$$

and for $1 \le k \le a$,

$$c_\pi(G[p],\,a-k) = (k!)^{-1}\frac{\partial^k}{\partial p^k}c_\pi(G[p],\,a). \tag{9}$$

Instead of the above statement 1 we demonstrate the validity of a somewhat more general result.

Theorem 1. *Eqs. (8) and (9) hold for $\pi = \mu$.*

Proof. In order to simplify the notation one may consider eq. (8) as a special case of eq. (9) for $k = 0$. Bearing in mind (4) we immediately see that Theorem 1 is equivalent to the claim that eq. (10) holds for $\pi = \mu$:

$$\pi(G[p]) = x^{pa-(a-b)}\sum_{k=0}^{a}(-1)^{a-k}\frac{x^{2k}}{k!}\frac{\partial^k}{\partial p^k}\sum_{k=0}^{a} c_\pi(G,\,m)p^{a-m}. \tag{10}$$

The above equality can be rewritten in a more compact form as

$$\pi(G[p]) = x^{pa-(a-b)} \sum_{k=0}^{a} (-1)^k \frac{x^{2k}}{k!} \frac{\partial^k}{\partial p^k} \left(\sqrt{-p}\right)^{a-b} \pi\left(G,\, \sqrt{-p}\right). \tag{11}$$

We prove (11) by induction on the number of edges of the graph G. Denote the number of edges of G by $E(G)$.

If $E(G) = 0$ then

$$\mu(G) = x^{a+b},$$
$$\mu(G[p]) = x^b \left(x^{p+1} - px^{p-1}\right)^a.$$

The fact that eqs. (8) and (9) are satisfied is readily verified. If $E(G) = 1$ then

$$\mu(G) = x^{a+b-2}(x^2 - 1),$$
$$\mu(G[p]) = x^{b-1} \left(x^{p+1} - px^{p-1}\right)^{a-1} \left[x^{p+2} - (p+1)x^p\right],$$

and eqs. (8) and (9) are satisfied again. Hence (11) holds for $E(G) = 0$ and $E(G) = 1$.

Suppose now that $E(G) = E_0$ and that (11) holds for all bipartite graphs with less than E_0 edges. We show that from this assumption it follows that (11) holds for the graph G as well.

Denote by $H_1 \cup H_2$ the graph whose components are H_1 and H_2 and recall that [**6**]

$$\mu\left(H_1 \cup H_2\right) = \mu\left(H_1\right) \mu\left(H_2\right). \tag{12}$$

If $H_1,\, H_2, \ldots,\, H_m$ are isomorphic graphs then $H_1 \cup H_2 \cup \cdots \cup H_m$ is denoted by mH_1.

As usual, K_n symbolizes the complete graph with n vertices and $\bar{K}_n$ its complement, i.e. the n–vertex graph without edges.

Let e be an edge of G connecting the vertices u and v and belonging to the circuits C_j, $j \in I(e)$. Then

$$G[p] - e = \{G - e\}[p],$$
$$G[p] - u - v = \bar{K}_p \cup \{G - u - v\}[p],$$
$$G[p] - C_j = n_j \bar{K}_p \cup \{G - C_j\}[p],$$

where $2n_j$ is the size of the circuit C_j. Applying (7) we have

$$\mu(G[p]) = \mu(\{G - e\}[p]) - x^p \mu(\{G - u - v\}[p]) - 2 \sum_{j \in i(e)} x^{pn_j} t_j \mu(\{G - C_j\}[p]). \tag{13}$$

Since both $G-e$, $G-u-v$ and $G-C_j$ have less edges than G, the induction hypothesis is applicable, viz.

$$\mu(\{G - e\}[p]) = x^{pa-(a-b)} \sum_{k=0}^{a} (-1)^k \frac{x^{2k}}{k!} \cdot \frac{\partial^k}{\partial p^k} \left(\sqrt{-p}\right)^{a-b} \mu\left(G - e,\, \sqrt{-p}\right), \tag{14}$$

(15)
$$\mu(\{G-u-v\}[p]) = x^{p(a-1)-(a-b)} \sum_{k=0}^{a-1} (-1)^k \frac{x^{2k}}{k!} \cdot \frac{\partial^k}{\partial p^k} \left(\sqrt{-p}\right)^{a-b} \mu\left(G-u-v,\, \sqrt{-p}\right),$$

(16)
$$\mu(\{G-C_j\}[p]) = x^{p(a-n_j)-(a-b)} \sum_{k=0}^{a-n_j} (-1)^k \frac{x^{2k}}{k!} \cdot \frac{\partial^k}{\partial p^k} \left(\sqrt{-p}\right)^{a-b} \mu\left(G-C_j,\, \sqrt{-p}\right).$$

Substituting (14)–(16) back into (13) we obtain

$$\mu(G[p]) = x^{pa-(a-b)} \sum_{k=0}^{a} (-1)^k \frac{x^{2k}}{k!} \frac{\partial^k}{\partial p^k} \left(\sqrt{-p}\right)^{a-b} \cdot \mu\left(G-e,\, \sqrt{-p}\right) - \mu\left(G-u-v,\, \sqrt{-p}\right) - 2 \sum_{j\in I(e)} t_j \mu\left(G-C_j,\, \sqrt{-p}\right),$$

which by another application of (7) yields eq. (11) for $\pi = \mu$.

3. A GENERALIZATION OF (1)

Theorem 2. *Eq. (1) holds for* $\pi = \mu$.

Proof follows by induction on the number of edges of G and is fully analogous to the proof of Theorem 1. One has to take into account (7) and (12) as well as the identities

$$G[R_1,\, R_2] - e = \{G-e\}[R_1,\, R_2],$$
$$G[R_1,\, R_2] - u - v = R_1^* \cup R_2^* \cup \{G-u-v\}[R_1,\, R_2],$$
$$G[R_1,\, R_2] - C_j = n_j R_1^* \cup n_j R_2^* \cup \{G-C_j\}[R_1,\, R_2].$$

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