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SYMMETRIC DERIVATIVES AND CONVEXITY CLASSES OF OVČARENKO

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The concept of v–derivative of real functions is used to study three classes of the Ovčarenko convexity [4]. The relationship of v–derivative to the Ovčarenko classes is the same as the relationship of the second symmetric (Schwarz) derivative to the class of the ordinary convexity. In this way, some theorems of Natanson [3] on behaviour of the second symmetric derivative and derivative numbers are generalized.

0. INTRODUCTION

In [**4**], I. E. Ovčarenko introduces the following concept of convexity of real functions.

Definition 1. *Let, for $p \in \mathbf{R}$, $v(x) = x + \frac{p}{3}x^3 + o(x^3)$. Then, we call f v–convex on $[a,\, b]$ if for every $x,\, x_1,\, x_2$ such that $a \le x_1 < x < x_2 \le b$, we can find $\delta > 0$ such that for $x_2 - x_1 < \delta$, the inequality*

$$(1) \qquad v(x_2 - x)f(x_1) + v(x_1 - x_2)f(x) + v(x - x_1)f(x_2) \ge 0$$

holds.

It is easy to show that for $p < 0$ and $\delta < \frac{\pi}{r}$ $(r \in \mathbf{R})$, v–convexity reduces on the convexity with respect to the Chebyshev system $\{\cos rx,\, \sin rx\}$, for $p = 0$ we have ordinary convexity, i.e. convexity with respect to $\{1,\, x\}$, while for $p > 0$, f is convex with respect to another Chebyshev system $\{\cosh rx,\, \sinh rx\}$. Thus, the class $V(v)$ of v–convex functions is the union of three classes: $V(\sin rx)$ – the class of trigonometric convexity, $V(x)$ – the class of ordinary convexity and $V(\sinh rx)$ – the class of hyperbolic convexity. Also, $V(v)$ can be considered as a special case of sub–F convexity ([**1**]). Namely, every $f \in V(v)$ can be majorized by

$$(2) \qquad L_{1,2}(x) = L(x_1,\, x_2;\, v;\, x) = \frac{v(x_2 - x)}{v(x_2 - x_1)}f(x_1) + \frac{v(x - x_1)}{v(x_2 - x_1)}f(x_2),$$

i.e. we have

$$f(x) \le L(x_1,\, x_2;\, v;\, x), \qquad x \in [x_1,\, x_2].$$

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In other words, $L_{1,2}$ are limit functions of the convex cone $V(v)$. It is known that such functions can be derived as the solution of the second order initial value problem

$$\begin{gathered} \frac{\mathrm{d}^2}{\mathrm{d}x^2}L(x) - 2pL(x) = 0, \\ L(x_1) = f(x_1), \quad L(x_2) = f(x_2). \end{gathered} \tag{3}$$

Note that, for $p = 0$, $L_{1,2}(x)$ is a straight line through the points $(x_1,\, f(x_1))$, $(x_2,\, f(x_2))$, while, for $p < 0$ we need the restriction $x_2 - x_1 < \frac{\pi}{r}$.

1. DERIVATIVES

It is known [**6**] that even the ordinary convex function may not have second derivative everywhere in its domain of convexity. According to Peixoto [**5**] things are not much better when we consider sub–F classes of convexity. So, we find it will make sense to study relationship of the Ovčarenko class, $V(v)$ with a more general type of derivatives like symmetric ones.

In this sense, let us recall

$$f^{[\prime]}(x) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} \tag{4}$$

$$f^{[\prime\prime]}(x) = \lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \tag{5}$$

where we assume the existence of the limits above. In [**3**] Natason studied a relationship between $f^{[\prime\prime]}$ and the class of convex functions. Here are some generalizations to the classes $v(\sin rx)$ and $V(\sinh rx)$.

Lemma 1. *Let $f\colon [a,\, b] \to \mathbf{R}$ be v–convex on $[a,\, b]$. Then $f^{[\prime]}(x)$ exists for any $x \in (a,\, b)$.*

Proof. According to ([**4**], p. 107), the one–sided derivatives $f'_-(x)$ and $f'_+(x)$ exist for any $x \in (a,\, b)$. On the other hand, by (4)

$$f^{[\prime]}(x) = \frac{1}{2}\lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h} - \frac{f(x) - f(x-h)}{h}\right) = \frac{1}{2}\left(f'_+(x) + f'_-(x)\right),$$

i.e. $f^{[\prime]}(x)$ exists. ¤

Definition 2. *Let $v(x) = x + \frac{p}{3}x^3 + o(x^3)$, and let us suppose that*

$$\lim_{h\to 0}\frac{1}{h^2}\left[f(x+h) - 2v'(h)f(x) + f(x-h)\right],$$

exists. Then, we call this limit the v*–derivative of* f *in* x *and denote it* $f^{[v]}(x)$ *or* $f^{[v(t)]}(x)$.

Lemma 2. *The derivative* $f^{[v]}(x)$ *exists if and only if* $f^{[\prime\prime]}(x)$ *exists. In this case*

$$f^{[v]}(x) = f^{[\prime\prime]}(x) - 2pf(x). \tag{6}$$

Proof. Let $f^{[\prime\prime]}(x)$ exist. Then, $f^{[\prime\prime]}(x) = \lim\limits_{h\to 0} \frac{1}{h^2}[f(x+h) - 2f(x)+ f(x-h)] = \lim\limits_{h\to 0} \left(\frac{f(x+h) - 2v'(h)f(x) + f(x-h)}{h^2} + 2\frac{v'(h)-1}{h^2} f(x) \right)$. By the definition of function v, we have $v'(h) = 1 + ph^2 + o(h^2)$, which gives

$$\lim_{h\to 0} \frac{2}{h^2} \left(v'(h) - 1\right) f(x) = 2pf(x),$$

i.e. (6) follows. On the other hand, let $f^{[v]}$ exist. Then,

$$f^{[v]}(x) = \lim_{h\to 0} \frac{1}{h^2} \left[f(x+h) - 2v'(h)f(x) + f(x-h)\right],$$

wherefrom we get (6). ¤

Now, we are ready to prove

Theorem 1. *Let* f *be defined and continuous on* $[a,\, b]$*. If* $f^{[v]}(x) = 0$*, for every* $x \in (a,\, b)$*, then*

$$f(x) = \mathrm{A}\sin rx + \mathrm{B}\cos rx, \qquad \text{for} \quad p < 0 \quad \text{and} \quad b - a < \frac{\pi}{r}, \tag{7}$$

$$f(x) = \mathrm{A}x + \mathrm{B}, \qquad \text{for} \quad p = 0, \tag{8}$$

$$f(x) = \mathrm{A}\sinh rx + \mathrm{B}\cosh rx, \qquad \text{for} \quad p > 0. \tag{9}$$

Proof. The proof is known for $p = 0$ ([**3**]). We shell carry out the proof for $p < 0$ and when $b - a < \frac{\pi}{r}$. The proof for $p > 0$ is quite similar. In this sense, consider for $\varepsilon > 0$ the function F given by

$$F(x) = f(x) + \varepsilon g(x), \tag{10}$$

where

$$g(x) = 1 - \frac{\cos \frac{r}{2}(2x - a - b)}{\cos \frac{r}{2}(b-a)}.$$

It is easy to see that $F(a) = L(a)$, $F(b) = L(b)$, where $L(x) = L(a,\, b;\, \sin rt;\, x)$, and

$$F^{[\sin rt]}(x) = \varepsilon r^2, \qquad x \in [a,\, b]. \tag{11}$$

Let us prove that the inequality

$$F(x) \le L(x) \tag{12}$$

holds everywhere on $[a,\, b]$. If we suppose that $F(x) \geq L(x)$ on $[a,\, b]$ then there exist the point $x_0 \in (a,\, b)$ such that for $h > 0$, $a < x_0 - h < x_0 + h < b$ we have

$$F(x_0 + h) - 2\cos rhF(x_0) + F(x_0 - h) < 0.$$

Dividing both sides by h, and letting $h \to 0_+$, we get $F^{[\sin rt]}(x_0) < 0$, which contradicts (11) because, against the supposition, $\varepsilon > 0$. Thus, x_0 does not exist and then, (12) is true. From (12) and (10) we have

$$f(x) - L(x) \leq -\varepsilon g(x), \qquad x \in [a,\, b],$$

which, in virtue of $m \leq g(x) \leq 0$, $x \in [a,\, b]$ $\left(m = -2\dfrac{\sin^2 \frac{r}{4}(b-a)}{\cos \frac{r}{2}(b-a)} \right)$, makes

$$f(x) - L(x) \leq \varepsilon |g(x)|. \tag{13}$$

Now, consider the function G defined by

$$G(x) = f(x) - \varepsilon g(x), \tag{14}$$

with the same $g(x)$. By (14) and $g(a) = g(b) = 0$ we have $G(a) = L(a)$, $G(b) = L(b)$. Also, by $g^{[\sin rt]}(x) = r^2$ and our presumption $f^{[\sin rt]}(x) = 0$, we get

$$G^{[\sin rt]}(x) = -\varepsilon r^2. \tag{15}$$

In the similar manner as above, one can prove

$$G(x) \geq L(x), \qquad x \in [a,\, b], \tag{16}$$

which, together with (14), gives

$$f(x) - L(x) \geq g(x) = -\varepsilon |g(x)|. \tag{17}$$

On the basis of (13) and (17) we get $|f(x) - L(x)| \leq \varepsilon |g(x)|$, which, if we keep in mind that ε is arbitrary, gives $f(x) = L(X)$, i.e. (7) holds.

As we have already noted, for $p > 0$ the proof is similar. Instead of (10), we need

$$F(x) = f(x) - L(x) - \varepsilon g(x), \qquad (\varepsilon > 0), \tag{18}$$

where $L(x) = L(a,\, b;\, \sinh rt;\, x)$ is given by (2), while

$$g(x) = 1 - \frac{\cosh \frac{r}{2}(2x - a - b)}{\cosh \frac{r}{2}(b - a)}. \tag{19}$$

Here, we first prove that $F(x) \le 0$. Then, $G(x) = -f(x) + L(x) - \varepsilon g(x)$, so

$$|f(x) - L(x)| \le \varepsilon g(x), \qquad x \in (a,\, b),$$

which, due to arbitrariness of ε, gives (9). ¤

2. CRITERIA OF v–CONVEXITY

It is clear that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) - 2pf(x) \ge 0, \tag{20}$$

is a criterion for v–convexity for any $f \in \mathbf{C}^2\,[a,\, b]$ i.e. (20) is necessary and sufficient for f to be v–convex. What is the criterion if f is not twice differentiable? To answer this question we will define v–derivative numbers.

Let $\mathbf{R}^\star = \mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$.

Definition 3. *We call $q(v;\, f)(x) \in \mathbf{R}^\star$ a v–derivative number of the function f in the point x, if there exists a sequence $(h_n)\,(n \in \mathbf{N})$ such that $h_n \ne 0$ $(n \in \mathbf{N})$, $\lim\limits_{n\to\infty} h_n = 0$, so that*

$$q(v;\, f)(x) = \lim_{n\to\infty} \frac{1}{h^2}\left[f(x+h_n) - 2v'(h_n)f(x) + f(x-h_n)\right], \tag{21}$$

where v is as in Definition 1.

Note that, for $v(x) = x$, $q(v;\, f)$ reduces to the second derivative number [3].

Theorem 2. *If the function f is defined on $[a,\, b]$, then, for every $x \in [a,\, b]$ the numbers $q(v;\, f)$ exist.*

Proof. Let $x_0 \in [a,\, b]$ and $(h_n)\,(n \in \mathbf{N})$ be a zero–sequence such that, for every $n \in \mathbf{N}$, $x_0 + h_n \in [a,\, b]$. Set

$$t_n = \frac{1}{h_n^2}\left[f(x_0+h_n) - 2v'(h_n)f(x_0) + f(x_0-h_n)\right], \qquad n \in \mathbf{N}.$$

If (t_n) is bounded sequence, then, on the basis of Bolzano–Weierstrass theorem, we can select a subsequence (t_{n_k}) which converges to the limit $q(v;\, f)(x)$. If (t_n) is unbounded from below (from above) then we can select a subsequence (t_{n_k}) which diverges to $+\infty$ (or $-\infty$), thus, $q(v;\, f)(x) = +\infty$ $(-\infty)$. ¤

Theorem 3. *Let f be defined on $[a,\, b]$. Then, $f^{[v]}(x)$ exists if and only if all v–derivative numbers $q(v;\, f)$ are equal.*

Proof. If $f^{[v]}(x)$ exists, then, obviously, all $q(v;\, f)$ are equal. On the other hand, if $q(v;\, f)(x) = q_0$, then, for every nontrivial zero–sequence (h_n), by (21)

$$\lim_{n\to\infty} \frac{1}{h_n^2}\left[f(x+h_n) - 2v'(h_n)f(x) + f(x-h_n)\right] = q_0. \tag{22}$$

Let us suppose that (22) is not true, i.e. that for at least one nontrivial zero–sequence (h_n), the sequence (s_n) given by

$$s_n = \frac{1}{h_n^2}\left[f(x+h_n) - 2v'(h_n)f(x) + f(x-h_n)\right],$$

does not converge to q_0. Let, $-\infty < q_0 < +\infty$. Then, $\varepsilon > 0$ exists so that infinitely many numbers s_n lie outside the interval $(q_0 - \varepsilon,\, q_0 + \varepsilon)$. From this set, a subsequence (s_{n_k}) can be selected so that (s_{n_k}) converges towards $s \in \mathbf{R}^\star$. This limit value, s, is exactly a v–derivative number of f in the point x and differs from q_0, which is again a contradiction. ¤

The following theorem gives a criterion for v–convexity.

Theorem 4. *f is v–convex in $[a,\, b]$ if and only if all $q(v;\, f)(x)$ are equal for all $x \in [a,\, b]$.*

Proof. Suppose $f \in K(v)$. Then, for any zero–sequence (h_n) $(n \in \mathbf{N})$, we have

$$f(x+h_n) - 2v'(h_n)f(x) + f(x-h_n) \geq 0,$$

so $q(v;\, f)$ are nonnegative, which establishes the necessity of the condition.

To prove that the condition is sufficient, we shall regard three classes separately. For $v(x) = x$ we can find the proof in ([**3**], p. 282). Let $v(x) = \sin rx$. Consider the function F given by (10). The condition of the theorem, together with $g^{[\sin rt]}(x) = r^2$ gives $q(\sin rt;\, F) \geq \varepsilon r^2 > 0$. In the proof of Theorem 1 it is shown that $F(x) \leq L(x)$, i.e. $f(x) \leq L(x) - \varepsilon g(x)$, with g as in this proof. When $\varepsilon \to 0_+$, we have

$$f(x) \leq L(x) = L(a,\, b;\, \sin rt;\, x),$$

i.e. $f \in K(\sin rt)$. In the case $v(x) = \sinh rx$, we shall define F as in (18) and $g(x)$ with (19). Now, $q(\sinh rx;\, F) \geq \varepsilon r^2$, so we have $F(x) \leq 0$ by the similar argument as in trigonometric case. Therefore, $f(x) \leq L(x) + \varepsilon g(x)$ and letting $\varepsilon \to 0_+$ we get $f(x) \leq L(a,\, b;\, \sinh rt;\, x)$, i.e. $f \in K(\sinh rx)$. ¤

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