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A NOTE ON HADWIGER'S CONJECTURE FOR LINE GRAPHS

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We prove that Hadwiger's conjecture for line graphs is true.

Consider a graph G without loops and multiple edges (a simple graph). The line graph $L(G)$ of G is the graph whose vertices are all the edges of G and whose two vertices are adjacent iff the corresponding edges are adjacent in G. Line graphs have been investigated by many authors [**1**]–[**6**] in various connections.

Analogously as in [**7**], we shall call the merging of two adjacent vertices of G, the deletion of loops generated by the merging of vertices and replacement of all thus generated multiple edges (if any) by edges, an elementary contraction of G. A graph G is contractible to a graph H if there exists a sequence of elementary contractions of vertices leading from G to H.

In the sequel K_m will denote the complete graph with m vertices. In [**8**], [**9**] HADWIGER formulated the conjecture that every connected graph with the chromatic number m is contractible to K_m.

For $m = 5$, HADWIGER'S conjecture has the four colour hypothesis as its immediate consequence. The inverse theorem has been proved by WAGNER [**10**]. Now, there exist proofs of the HADWIGER conjecture for $m \leq 4$ by DIRAC [**11**] and for $m = 5$ by HAKEN and APPEL in collaboration with KOCH [**12**], as well as proofs of several weaker results; e.g. ČULÍK in [**13**] has proved the HADWIGER conjecture for G such that: (1) G is a simple graph; (2) adding an edge connecting vertices which are not adjacent in G leads to a graph with the vertex chromatic number greater than that of G.

The present note deals with another partial result, namely the proof of the HADWIGER conjecture for the line graphs.

Let $\Delta(G)$ be the greatest degree of vertices of G and $\chi_1(G)$ be the edge chromatic number of G.

G will be called r–critical as in [**14**], if

(a) G is connected,

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(b) $\Delta(G) = r - 1$,

(c) $\chi_1(G) = r$,

(d) the edge chromatic number of any graph obtained from G by deleting an arbitrarily selected edge is $r - 1$.

The following statements are proved in [**15**]:

Theorem 1. *For any graph G with $\Delta(G) = m$, $m \leq \chi_1(G) \leq m + 1$.*

Lemma 1. *No r–critical graph with $r \geq 3$ has an articulation.*

Lemma 2. *Let G be a graph with $\Delta(G) = m - 1$ and $\chi_1(G) = m$. Then for any integer t such that $3 \leq t \leq m$ there exists a t–critical subgraph of G.*

The proof of this lemma can be found in [**14**].

The following theorem asserts the validity of the Hadwiger conjecture for line graphs.

Theorem 2. *Any connected line graph with the chromatic number m is contractible to K_m.*

Proof. For any G, $\chi_1(G) = \chi_0(L(G))$. (Here $\chi_0(T)$ is the vertices chromatic number of the graph T). According to Theorem 1, $\Delta(G) \leq \chi_1(G) \leq \Delta(G) + 1$. Let us distinguish the following two cases:

1° $\chi_1(G) = \Delta(G) = m$. Then there exists a subgraph of $L(G)$ isomorphic to K_m; this proves the theorem.

2° $\chi_1(G) = \Delta(G) = 1 = m$. Without loss of generality (cf. Lemma 2) we may suppose that G is m–critical with $\Delta(G) = m-1$. Let x be one of the vertices of G with degree $m - 1$ and $x_1, x_2, \dots, x_{m-1}$ the distinct vertices adjacent to x. By definition of $L(G)$, any edge xx_i in G has a unique corresponding vertex y_i $(1 \leq i \leq m-1)$ in $L(G)$. Evidently the subhraph of $L(G)$ iduced by the set $y_1, y_2, \dots, y_{m-1}$ is isomorphic to K_{m-1}. Let

$$\mathcal{B} = V(L(G)) - \{y_1, y_2, \dots, y_{m-1}\},$$

where $V(P)$ is the set of all vertices of the graph P. It is easy to show that $\mathcal{B} = \emptyset$ iff G is a star. By Lemma 1, a star is not an m–critical graph; thus it is sufficient to consider $\mathcal{B} = \emptyset$. Again let us distinguish two alternatives:

(a) $|\mathcal{B}| = 1$. By the hypothesis, $\chi_1(G) = \Delta(G) + 1 = m$. Since there are no 2–critical graphs, it is enough to investigate the case $m \geq 4$.

(b) $|\mathcal{B}| \geq 2$. By Lemma 1, G has no articulation and by the hypothesis it is connected; therefore for every i, j $(i \neq j)$ there exists a path connecting the vertices x_i and x_j which does not pass through x. Therefore in $L(G)$, any vertex y_i for $i = 1, 2, \dots, m - 1$ is adjacent to at least one vertex of $\mathcal{B}$. Thus by Lemma 1, the subgraph of $L(G)$ induced by all vertices from $\mathcal{B}$ is connected. By a sequence of elementary contractions of vertices of $\mathcal{B}$ in $L(G)$ we obtain one vertex adjacent to every y_i $(1 \leq i \leq m - 1)$; thus the resulting graph is isomorphic to K_m.

In both alternatives, we have shown that the graph $L(G)$ with the chromatic number m is contractible to K_m. This completes the proof of Theorem 2.

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