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THE C–FUNCTION OF E. W. BARNES

Luciana Lupaş

We consider some functional equations involving the gamma function, e.g. (1) and (2). We prove that such equations have unique solutions in the class of logarithmically convex functions. Such solutions give rise to some new special functions.

0. INTRODUCTION

The aim of this paper is to find the solutions, with a certain shape, of the functional equations

$$G(x+1) = \Gamma(x)G(x), \qquad x \in \mathbf{R}_+ := (0, +\infty) \tag{1}$$

or

$$F(x+1) = x^x F(x), \qquad x \in \mathbf{R}_+. \tag{2}$$

These equations will be solved in the set LK_2 of log-convex (logarithmically convex) functions of the second order on $\mathbf{R}_+$. More precisely, a function $f\colon \mathbf{R}_+ \to \mathbf{R}_+$ belongs to the class LK_2 if and only if

$$[x_1,\, x_2,\, x_3,\, x_4;\, \ln f] > 0$$

for any system $x_1,\, x_2,\, x_3,\, x_4$ of distinct points in $\mathbf{R}_+$; by $[x_1,\, x_2,\, x_3,\, x_4;\, .]$ we denote the divided difference at the specified points. We will show that the functional equation (1) has an unique solution G in LK_2, with $G(1) = 1$; a similar result is established for (2). This leads to axiomatic definitions of certain special functions. For instance, the solution G, $G \in \mathrm{LK}_2$, $G(1) = 1$, of the equation (1) coincides with the so-called "G–function" studied by E. W. BARNES [**3**], [**9**]. We use this characterization as the basis for the investigation of the G–function.

Our work is motivated by the results from [**1**], [**2**], [**6**]. We note that W. KRULL [**5**] has considered the functional equation $f(x+1) = g(x)f(x)$, $x \in \mathbf{R}_+$, where

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$g\colon \mathbf{R}_+ \to \mathbf{R}_+$ is a prescribed function which is continuous, log-convex on $(a, +\infty)$, $a > 0$, and where $\lim_{x\to\infty} \frac{g(x+1)}{g(x)} = 1$. If we select $g(x) = \Gamma(x)$ or $g(x) = x^x$, then the last condition is not verified; therefore we can not apply the work of W. KRULL.

1. THE FUNCTIONAL EQUATION $G(X+1) = \Gamma(X)G(X)$

Lemma 1.1. *There exists at least one function $G\colon \mathbf{R}_+ \to \mathbf{R}_+$ satisfying the conditions*

(*i*) $$G(x+1) = \Gamma(x)G(x), \qquad x \in \mathbf{R}_+,$$

(*ii*) $$G \in \mathrm{LK}_2,$$

(*iii*) $$G(1) = 1.$$

Proof. Let $G\colon \mathbf{R}_+ \to \mathbf{R}_+$ be defined by

(3) $$G(x) = \exp\left\{\frac{x}{2}\ln 2\pi - \frac{x(x-1)}{2} + (x-1)\ln\Gamma(x) - \int_0^x \ln\Gamma(t)\mathrm{d}t\right\}.$$

In wiew of "RAABE formula"

$$\int_x^{x+1} \ln\Gamma(t)\mathrm{d}t = x\ln x - x + \frac{1}{2}\ln 2\pi, \qquad x > 0,$$

we find that $G(x+1) = \Gamma(x)G(x)$ and $G(1) = 1$. Moreover, if $p := \ln G$ then

$$p'''(x) = 2[\ln\Gamma(x)]'' + (x-1)[\ln\Gamma(x)]''' = 2\sum_{k=0}^{\infty} \frac{k+1}{(x+k)^3},$$

that is p'' is positive on $\mathbf{R}_+$; therefore $G \in \mathrm{LK}_2$.

Theorem 1.2. *The functional equation $G(x+1) = \Gamma(x)G(x)$, $x \in \mathbf{R}_+$, $G(1) = 1$, has an unique solution $G\colon \mathbf{R}_+ \to \mathbf{R}_+$ which is log-convex of the second order on $\mathbf{R}_+$.*

Proof. On the set of natural numbers, any solution G is determined by $G(1) = 1$, $G(n+1) = \Gamma(1)\Gamma(2)\cdots\Gamma(n)$, $n = 1, 2, \ldots$.

Likewise, for any natural number n one has

(4) $$G(n+x) = G(x)\prod_{k=1}^{n}\Gamma(x+k-1), \qquad x \in \mathbf{R}_+.$$

If we select $n = [x]$, $x = [x] + \{x\}$, then (4) implies

(5) $$G(x) = G(1+\{x\})\prod_{k=2}^{[x]}\Gamma(\{x\}+k-1), \qquad x > 1.$$

Let $x \in (0,1]$; from $[n,\, n+x,\, n+1,\, n+x+1;\, \ln G] > 0$ we find

$$(6)\qquad [G(n)]^{1-x}[G(n+1)]^{1+x} \le [G(n+x)]^{1+x}[G(n+x+1)]^{1-x}, \qquad n = 1,2,\dots ,$$

with equality if and only if $x = 1$. Since $G(y+1) = \Gamma(y)G(y)$, we have

$$(7)\qquad G(n+x) \ge G(n+1)[\Gamma(n)\Gamma(n+x)]^{\frac{x-1}{2}}, \qquad x \in (0,1].$$

Now we use the convexity of the second order of $\ln G$ at the points $n+x$, $n+1$, $n+x+1$, $n+2$; the inequality

$$[n+x,\, n+1,\, n+x+1,\, n+2;\, \ln G] > 0$$

gives

$$(8)\qquad G(n+x) \le G(n+1)[\Gamma(n+1)]^{\frac{x}{2}}[\Gamma(n+x)]^{\frac{x-2}{2}}.$$

Applying (4), (7) and (8) we derive the inequalities

$$(9)\qquad m_n(x) \le \frac{G(x)}{G_n(x)} \le 1, \qquad (x \in (0,1];\quad n = 1,2,\dots),$$

where

$$m_n(x) = \left[\frac{\Gamma(n+x)}{\Gamma(n+1)} n^{1-x}\right]^{\frac{1}{2}}$$

and for $x \in \mathbf{R}_+$

$$(10)\qquad G_n(x) = \frac{\Gamma(1)\Gamma(2)\cdots\Gamma(n)[\Gamma(n+1)\Gamma(n+x)]^{\frac{x}{2}}}{\Gamma(x)\Gamma(x+1)\cdots\Gamma(x+n)}.$$

But Γ is log-convex (of the first order) on $\mathbf{R}_+$; therefore $[n+x,\, n+1,\, n+2;\, \ln\Gamma] > 0$ which is the same as

$$m_n(x) \ge \left(\frac{n}{n+1}\right)^{\frac{1-x}{2}}.$$

From (9) we conclude that the sequence $(G_n(x))$, $n = 1,2,\dots$, $x \in (0,1]$, converges and

$$(11)\qquad \lim_{n\to\infty} G_n(x) = G(x).$$

Now we show that (11) holds for any $x \in \mathbf{R}_+$. From (10) one finds

$$(12)\qquad G_n(x+1) = \Gamma(x)G_n(x)[a_n(x)]^{\frac{1}{2}}, \qquad x \in \mathbf{R}_+,$$

with

$$a_n(x) = \frac{\Gamma(n+1)}{\Gamma(n+x)}(n+x)^{x-1}.$$

Using inequalities for gamma function (see [**7**], p. 286) we obtain

$$e^{-\frac{1}{4n}} < a_n(x) \le 1, \qquad x \in (0,1]; \qquad n = 1, 2, \dots .$$

Let us suppose that x is an arbitrary point in $[1, +\infty)$; from (5) and (10) we observe that

$$G_n(x) = G(x)\frac{G_n(1+\{x\})}{G(1+\{x\})}[P_n(x)]^{\frac{1}{2}} \tag{13}$$

where

$$P_n(x) = \prod_{k=2}^{[x]} a_n(\{x\}+k-1).$$

By means of STIRLING formula

$$1 \le P_n(x) < \exp\left(\frac{x^3}{4n^2}\right), \qquad x \in [1, +\infty); \qquad n = 1, 2, \dots .$$

The equality (13) and the above inequalities show us that any function $G\colon \mathbf{R}_+ \to \mathbf{R}_+$ which verifies $G(x+1) = \Gamma(x)G(x)$, $x \in \mathbf{R}_+$, $G(1) = 1$, $G \in \mathrm{LK}_2$, is well-determined by

$$G(x) = \lim_{n\to\infty} \frac{\Gamma(1)\Gamma(2)\cdots\Gamma(n)[\Gamma(n+1)\Gamma(n+x)]^{\frac{x}{2}}}{\Gamma(x)\Gamma(x+1)\cdots\Gamma(x+n)}, \qquad x \in \mathbf{R}_+.$$

This completes the proof of our theorem.

In the following we shall denote by $G\colon \mathbf{R}_+ \to \mathbf{R}_+$ the solution in LK_2 of the equation $G(x+1) = \Gamma(x)G(x)$, $G(1) = 1$. Also, we say that G is the BARNES function. In the same time we introduce the notation

$$[\Gamma(n)]! := \Gamma(1)\Gamma(2)\cdots\Gamma(n).$$

Theorem 1.3. *The* BARNES *function verifies the equalities*

$$\begin{aligned} G(x) &= \exp\left\{\frac{x}{2}\ln 2\pi - \frac{x(x-1)}{2} + (x-1)\ln\Gamma(x) - \int_0^x \ln\Gamma(t)\mathrm{d}t\right\} \\ &= \lim_{n\to\infty} \frac{[\Gamma(n)]!\,[\Gamma(n+1)\Gamma(n+x)]^{\frac{x}{2}}}{\Gamma(x)\Gamma(x+1)\cdots\Gamma(x+n)} = A(x), \qquad x \in \mathbf{R}_+, \end{aligned} \tag{15}$$

where

$$A(x) = (2\pi)^{\frac{x-1}{2}} \exp\left[\frac{x(1-x)}{2} - \frac{\gamma(x-1)^2}{2}\right] \prod_{k=1}^{\infty} Q_k(x)$$

with

$$Q_k(x) = \left(1 + \frac{x-1}{k}\right)^k \exp\left[1 - x + \frac{(1-x)}{2k}\right]$$

and γ being the Euler *constant.*

Proof. The first equality is motivated by (3) and by the above Theorem 1.2. The equality $G = A$ was established by Alexeiewski ([**9**], p. 264); more precisely, it was shown that $A: \mathbf{R}_+ \to \mathbf{R}_+$ is an integral function which satisfies $A(1) = 1$, $A(x+1) = \Gamma(x)A(x)$. On the other hand $A \in \mathrm{LK}_2$; therefore, Theorem 1.2 enables us to assert that $A = G$.

2. THE FUNCTIONAL EQUATION $F(X+1) = X^X F(X)$

Lemma 2.1. *There exists a function $f: \mathbf{R}_+ \to \mathbf{R}_+$ with the properties*

$$\text{(i)} \qquad F(x+1) = x^x F(x), \qquad x \in \mathbf{R}_+,$$

$$\text{(ii)} \qquad F \in \mathrm{LK}_2,$$

$$\text{(iii)} \qquad F(1) = 1.$$

Proof. If

$$F(x) = \exp\left\{\frac{x(x-1)}{2} - \frac{x}{2}\ln 2\pi + \int_0^x \ln\Gamma(t)\mathrm{d}t\right\}, \qquad x \in \mathbf{R}_+, \tag{16}$$

then it can be easily shown that $F(x+1) = x^x F(x)$, $F(1) = 1$; also

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}(\ln F) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\ln \Gamma) > 0,$$

that is $F \in \mathrm{LK}_2$.

Theorem 2.2. *The equation $F(x+1) = x^x F(x)$, $x \in \mathbf{R}_+$, $F(1) = 1$, has an unique solution $F: \mathbf{R}_+ \to \mathbf{R}_+$ which belongs to the class* LK_2.

Proof. Let $x \in (0,1)$; from the fact that the inequalities $[n,\, n+x,\, n+1,\, n+x+1;\, \ln F] > 0$ and $[n+x,\, n+1,\, n+x+1,\, n+2;\, \ln F] > 0$ are verified, we have

$$\begin{aligned} &F(n+1)n^{\frac{n(x-1)}{2}}(n+x)^{\frac{(n+x)(x-1)}{2}} \le F(n+x) \le \\ &\le F(n+1)(n+1)^{\frac{x(n+1)}{2}}(n+x)^{\frac{(x-2)(n+x)}{2}}. \end{aligned} \tag{17}$$

But

$$F(n+x) = F(x)\prod_{k=1}^{n}(x+k-1)^{x+k-1}. \tag{18}$$

Next, define $F_n: \mathbf{R}_+ \to \mathbf{R}_+$ by

$$F_n(x) = \frac{1^1 2^2 \cdots n^n (n+1)^{\frac{x(n+1)}{2}}(n+x)^{\frac{x(n+x)}{2}}}{x^x(x+1)^{x+1}\cdots(x+n)^{x+n}}.$$

From (17) and (18) we obtain

$$q_n(x) \le \frac{F(x)}{F_n(x)} \le 1, \qquad x \in (0,1]; \qquad n = 1, 2, \dots ,$$

where

$$q_n(x) = \left[\frac{(n+x)^{n+x}}{(n+1)^{x(n+1)} n^{n(1-x)}} \right]^{\frac{1}{2}} .$$

Taking into account that $[n-1,\, n,\, n+x;\, t \ln t] > 0$, $n \ge 2$, we find that

$$e^{-\frac{1}{n}} < \frac{F(x)}{F_n(x)} \le 1, \qquad x \in (0,1].$$

Therefore $(F_n(x))_{n=1}^{\infty}$ converges on $(0,1]$ to $F(x)$; likewise $F_n(1) = 1$. On the other hand

$$F_n(x+1) = x^x F_n(x) [C_n(x)]^{\frac{1}{2}} \tag{19}$$

with

$$C_n(x) = \frac{(n+1)^{n+1} (n+x+1)^{(x-1)(n+x+1)}}{(n+x)^{x(n+x)}}, \qquad x \in \mathbf{R}_+ .$$

For $x \in (0,1]$ one has

$$e^{-\frac{1}{n}} < C_n(x) \le 1$$

while for $x \in (1, +\infty)$

$$1 < C_n(x) < e^{\frac{x^2}{n}} .$$

Now we observe from (18) and (19) that

$$F_n(x) = F(x) \frac{n}{F(1+x)} \prod_{j=2} [C_n(\{x\} + j - 1)], \qquad x \ge 2.$$

Then the above inequalities give $F(x) = \lim_{n\to\infty} F_n(x)$ for every $x \in (1, +\infty)$. In conclusion, the equality

$$F(x) = \lim_{n\to\infty} \frac{[n^n]!\, (n+1)^{\frac{x(n+1)}{2}} (n+x)^{\frac{x(n+x)}{2}}}{x^x (x+1)^{x+1} \cdots (x+n)^{x+n}} \tag{20}$$

where $[n^n]! = 1^1 2^2 \cdots n^n$, is verified for all positive x; this completes the proof of the theorem.

The equalities (15) and (20) enable us to prove the following proposition.

Corollary 2.3. *If* $F\colon \mathbf{R}_+ \to \mathbf{R}_+$, $F(1) = 1$, *is the solution in* LK_2 *of the functional equation* $F(x+1) = x^x F(x)$, $x \in \mathbf{R}$, *then*

$$F(x) G(x) = \Gamma^{x-1}(x), \qquad x \in \mathbf{R}_+ ,$$

where $G\colon \mathbf{R}_+ \to \mathbf{R}_+$ *is the* Barnes *function.*

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ADDED IN PROOF

The referee kindly informed us that there exist certain additions to the results exposed here. He gave the following references:

L. Bendersky: *Sur la fonction gamma généralisée.* Acta Math., **61** (1933), 263-322.

F. John: *Special solutions of certain difference equations.* Acta Math., **71** (1939), 175-189.

Facultatea de Mecanică,
Str. I. Raţiu 7,
2400 Sibiu,
Romania

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