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743. ON ONE PROPERTY EQUIVALENT TO THE UNIQUENESS OF THE SOLUTION OF THE CHEBYSHEV'S BEST APPROXIMATION PROBLEM IN A NORMED LINEAR REAL VECTOR SPACE*

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Let V be a normed linear real vector space. P. L. Chebyshev solved the problem of the best approximation of a function with the linear combination of the given functions. It is well known that the same property values in any space V (See [1] p. p. 68—69). In the present paper it is proved that for any two points of the space V and every finite-dimensional plane, which together with each of them generates the same sub-space, the following statement is valid: For each segment connecting one of these points with the corresponding point of the plane, where a minimum distance is achieved, there is a parallel segment corresponding to another point (Theorem 1). Besides, we consider that the segment turned-point is parallel to any segment. Then it is shown that the property of uniqueness of this point of that plane where the minimum is reached, is equivalent to the property that the mentioned segments are always parallel (Consequence 1). In such a way we obtain that the space V having the property of uniqueness of the solution of Chebyshev's best approximation problem, enables us to introduce the notion of the projection (Definition 1), which, in a certain way, retains some properties of the orthonormal projections in the Hilbert space (i. e. the uniqueness of the projection on the plane and the parallelism of segments connecting them with originals of projections). Finally, an example of such a space which is neither Banach's nor pre-Hilbert's is presented.

1. The introductory elements are already given in abstract. Therefore we shall proceed with the results directly.

Theorem 1. *Let* $x_1 - x_0, \ldots, x_n - x_0$ *be a set of linearly independent vectors of a normed linear real vector space* **V**. *Let then*

$$Z(y) = \{z_y \in \mathbf{V} \mid \| y - z_y \| = \min_{z \in x_0 + L(x_1 - x_0, \ldots, x_n - x_0)} \| y - z \| \}.$$

If for vectors $y_1, y_2 \in \mathbf{V}$ *the following is valid*

$$L(y_1 - x_0, x_1 - x_0, \ldots, x_n - x_0) = L(y_2 - x_0, x_1 - x_0, \ldots, x_n - x_0),$$

then for each $z_{y_1} \in Z(y_1)$ *there is some* $z_{y_2} \in Z(y_2)$ *and a scalar* λ, *so that at least one of the following relations*

$$y_1 - z_{y_1} = \lambda \cdot (y_2 - z_{y_2}), \quad y_2 - z_{y_2} = 0$$

is fulfilled.

Proof. Let $y_1 \in x_0 + L(x_1 - x_0, \ldots, x_n - x_0)$. Then $y_1 = z_{y_1}$, hence for $\lambda = 0$ the equality $y_1 - z_{y_1} = 0 = 0 \cdot (y_2 - z_{y_2})$ is valid. In the same way, if we have

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$y_2 \in x_0 + L(x_1 - x_0, \ldots, x_n - x_0)$, then the second of the above two relations follows.

Let us suppose now that $y_1, y_2 \notin x_0 + L(x_1 - x_0, \ldots, x_n - x_0)$.

Let $z_{y_1} = x_0 + \varphi_1(x_1 - x_0) + \cdots + \varphi_n(x_n - x_0)$ be an arbitrary vector from $Z(y_1)$. Then, according to the assumptions of the theorem,

$$y_1 - z_{y_1} = y_1 - x_0 - \varphi_1(x_1 - x_0) - \cdots - \varphi(x_n - x_0)$$
$$\in L(y_2 - x_0, x_1 - x_0, \ldots, x_n - x_0),$$

so that $\lambda \neq 0$ and scalars $\psi_1', \ldots, \psi_n'$ exist, so that

$$(1) \qquad \begin{aligned} &y_1 - x_0 - \varphi_1(x_1 - x_0) - \cdots - \varphi_n(x_n - x_0) \\ &= \lambda(y_2 - x_0) - \psi_1'(x_1 - x_0) - \cdots - \psi_n'(x_n - x_0). \end{aligned}$$

Indeed, if it was $\lambda = 0$, then from (1) it stems that

$$y_1 = x_0 + (\varphi_1 - \psi_1')(x_1 - x_0) + \cdots + (\varphi_n - \psi_n')(x_n - x_0)$$
$$\in x_0 + L(x_1 - x_0, \ldots, x_n - x_0),$$

which is contrary to the supposition. Further on, the vector $y_2 - x_0$ is linearly independent with respect to vectors $x_1 - x_0, \ldots, x_n - x_0$ so that scalars $\lambda, \psi_1', \ldots, \psi_n'$ are uniquely determined. Namely, if

$$y_2 - x_0 \in L(x_1 - x_0, \ldots, x_n - x_0),$$

then $y_2 \in x_0 + L(x_1 - x_0, \ldots, x_n - x_0)$, which is contrary to the supposition. If we put now $\psi_i = \frac{\psi_i'}{\lambda}$ $(i = 1, \ldots, n)$, from (1) it stems

$$(2) \qquad \begin{aligned} &y_1 - x_0 - \varphi_1(x_1 - x_0) - \cdots - \varphi_n(x_n - x_0) \\ &= \lambda \cdot (y_2 - x_0 - \psi_1(x_1 - x_0) - \cdots - \psi_n(x_n - x_0)). \end{aligned}$$

Let, further, $z_{y_2} = x_0 + \zeta_1(x_1 - x_0) + \cdots + \zeta_n(x_n - x_0)$ be any vector from $Z(y_2)$. In the same way as in (2) we get

$$(3) \qquad \begin{aligned} &y_2 - x_0 - \zeta_1(x_1 - x_0) - \cdots - \zeta_n(x_n - x_0) \\ &= \mu(y_1 - x_0 - \theta_1(x_1 - x_0) - \cdots - \theta_n(x_n - x_0)), \end{aligned}$$

where we have $\mu \neq 0$. Since from (2) it follows

$$(4) \qquad y_1 - x_0 = \lambda \cdot (y_2 - x_0) + (\varphi_1 - \lambda \cdot \psi_1)(x_1 - x_0) + \cdots + (\varphi_n - \lambda \cdot \psi_n)(x_n - x_0),$$

and from (3) it stems that

$$(5) \qquad y_1 - x_0 = \frac{1}{\mu}(y_2 - x_0) + \left(\theta_1 - \frac{1}{\mu}\zeta_1\right)(x_1 - x_0) + \cdots + \left(\theta_n - \frac{1}{\mu}\zeta_n\right)(x_n - x_0),$$

due to the uniqueness of representation $y_1 - x_0$ in $L(y_2 - x_0, x_1 - x_0, \ldots, x_n - x_0)$, from (4) and (5) it follows that $\frac{1}{\mu} = \lambda$, i.e.

$$(6) \qquad \lambda \cdot \mu = 1.$$

Let us suppose that $x_0+\psi_1(x_1-x_0)+\cdots+\psi_n(x_n-x_0)\notin Z(y_2)$. Then from (2) it stems

$$\|y_1-x_0-\varphi_1(x_1-x_0)-\cdots-\varphi_n(x_n-x_0)\|$$
$$>|\lambda|\,\|y_2-x_0-\zeta_1(x_1-x_0)-\cdots-\zeta_n(x_n-x_0)\|,$$

and having in view (3) we get

$$\|y_1-x_0-\varphi_1(x_1-x_0)-\cdots-\varphi_n(x_n-x_0)\|$$
$$>|\lambda|\,|\mu|\,\|y_1-x_0-\theta_1(x_1-x_0-\cdots-\theta_n(x_n-x_0)\|.$$

Herefrom, using (6) we have

$$\|y_1-x_0-\varphi_1(x_1-x_0)-\cdots-\varphi_n(x_n-x_0)\|$$
$$>\|y_1-x_0-\theta_1(x_1-x_0)-\cdots-\theta_n(x_n-x_0)\|,$$

which is impossible, since

$$x_0+\varphi_1(x_1-x_0)+\cdots+\varphi_n(x_n-x_0)\in Z(y_1).$$

Thus

$$x_0+\psi_1(x_1-x_0)+\cdots+\psi_n(x_n-x_0)\in Z(y_2),$$

so that on the basis of (2) we have $y_1-z_{y_1}=\lambda\cdot(y_2-z_{y_2})$, which completes the proof of the theorem.

Theorem 2. *Let $x_1-x_0,\ldots,x_n-x_0$ be a set of linearly independent vectors of a normed linear real vector space* $\mathbf{V}$. *Let*

$$Z(y)=\{z_y\in\mathbf{V}\mid \|y-z_y\|=\min_{z\in x_0+L(x_1-x_0,\ldots,x_n-x_0)}\|y-z\|\}.$$

If for any two vectors $y_1, y_2\in\mathbf{V}$, for which

$$L(y_1-x_0,x_1-x_0,\ldots,x_n-x_0)=L(y_2-x_0,x_1-x_0,\ldots,x_n-x_0)$$

is valid, we have that for any two vectors $z_{y_1}\in Z(y_1)$, $z_{y_2}\in Z(y_2)$ there is a scalar λ, so that at least one of the relations $y_1-z_{y_1}=\lambda\cdot(y_2-z_{y_2})$ or $y_2-z_{y_2}=0$ is satisfied, then for each $y\in\mathbf{V}$, $Z(y)$ has exactly one element.

Proof. It is obvious that there is at least one $z_y\in Z(y)$. Let us assume that there are at least two different vectors z_y' and z_y'' from $Z(y)$. Since

$$L(y-x_0,x_1-x_0,\ldots,x_n-x_0)=L(y-x_0,x_1-x_0,\ldots,x_n-x_0)$$

on the basis of the supposition of the theorem we have

$$y-z_y'=\lambda\cdot(y-z_y''). \tag{7}$$

If $y\in x_0+L(x_1-x_0,\ldots,x_n-x_0)$, then $z_y'=y=z_y''$.

On the other hand, if $y\notin x_0+L(x_1-x_0,\ldots,x_n-x_0)$, then from (7) it follows $(1-\lambda)\,y=z_y'-\lambda\cdot z_y''$, so that it must be either $1-\lambda=0$ or $y=0$. In the case that $\lambda=1$, then from (7) it stems $z_y'=z_y''$. If $y=0$, on the basis of definition of $Z(y)$ it follows $\|z_y'\|=\|z_y''\|$, wherefrom, having in view (7), we have

$$\|z_y'\|=|\lambda|\;\|z_y''\|,$$

i. e. $|\lambda|=1$. The case $\lambda=1$ is already considered, case $\lambda=-1$, $x_0 \neq 0$ yields from the assumption

$$z_y' = x_0 + \varphi_1 (x_1 - x_0) + \cdots + \varphi_n (x_n - x_0),$$

which means

$$z_y'' = -z_y' = -x_0 - \varphi_1 (x_1 - x_0) - \cdots - \varphi_n (x_n - x_0),$$

so that $z_y'' \notin x_0 + L(x_1 - x_0, \ldots, x_n - x_0)$, because its sum of coefficients besides $x_0, x_1, \ldots, x_n$ is not 1 but -1. If $\lambda=-1$, $x_0=0$, then it is already considered, since $y=0 \in L(x_1, \ldots, x_n)$. Thus, it must be $z_y' = z_y''$, so that the statement is completed.

From theorem 1 and Theorem 2 it stems

Consequence 1. *Let $x_1 - x_0, \ldots, x_n - x_0$ be a set of linearly independent vectors of a normed linear real vector space* $\mathbf{V}$ *and*

$$Z(y) = \{z_y \in \mathbf{V} \mid \|y - z_y\| = \min_{z \in x_0 + L(x_1 - x_0, \ldots, x_n - x_0)} \|y - z\|\}.$$

Let us denote by (i) *and* (ii) *the following statements:*

(i) *The set $Z(y)$ has exactly one element.*

(ii) *For any two vectors $y_1, y_2 \in \mathbf{V}$ which satisfy*

$$L(y_1 - x_0, x_1 - x_0, \ldots, x_n - x_0) = L(y_2 - x_0, x_1 - x_0, \ldots, x_n - x_0)$$

we have that for any two vectors $z_{y_1} \in Z(y_1)$, $z_{y_2} \in Z(y_2)$ there is a scalar λ, so that at least one of the following relations $y_1 - z_{y_1} = \lambda \cdot (y_2 - z_{y_2})$ or $y_2 - z_{y_2} = 0$ is satisfied.

Then (i) $\Leftrightarrow$ (ii).

2. Consequence 1 enables us to separate and to describe simply a subclass of the class of normed linear real vector spaces, making possible the consideration of some geometrical problems over a finite-dimensional plane. That is why we shall introduce.

Definition 1. *A normed linear real vector space* $\mathbf{V}$ *where for each $y \in \mathbf{V}$ there is exactly one vector z_y so that*

$$\|y - z_y\| = \min_{z \in x_0 + L(x_1 - x_0, \ldots, x_n - x_0)} \|y - z\|,$$

is true for each set of linearly independent vectors $x_1 - x_0, \ldots, x_n - x_0$ in $\mathbf{V}$, *is called* ***filled.*** *Vector z_y is then called* ***the filled projection*** *of vector y in $x_0 + L(x_1 - x_0, \ldots, x_n - x_0)$.*

That definition 1 has make a sence, we shall demonstrate with the following two examples:

EXAMPLE 1. $\mathbf{C}[0,1]$ is normed linear real vector space but not a filled one. (The solution of the CHEBYSHEV's best approximation problem is not unique).

EXAMPLE 2. Let $\mathbf{W}_p$ $(p>1,\ p \neq 2)$ be a set of all vectors in l_p with only a finite number of coordinates different from zero. Set $\mathbf{W}_p$ with the norm induced by l_p is the filled space which is neither BANACH nor pre-HILBERT space.

Namely, space $l_p\,(p>1)$ is strictly normed space (for each pair of vectors $u\neq 0$, $v\neq 0$ equality $\|u+v\|=\|u\|+\|v\|$ holds only if $u=\alpha\cdot v$, $\alpha>0$) and therefore a filled one (See [1] p. p. 69—70). It is obvious that W_p is normed linear subspace of l_p. The sequence $x_n=(\xi_{nk})$, where

$$\xi_{nk}=\begin{cases} 2^{-k}, & k=1,\ldots, n \\ 0 & , k=n, n+1,\ldots \end{cases}$$

is CAUCHY's sequence in W_p but does not converge in W_p. So W_p is not BANACH space. Vectors (1, 0, . . . , 0, . . .) and (0, 1, 0, . . . , 0, . . .) from W_p do not satisfy the parallelogram relation. Consequently W_p is not pre-HILBERT space.

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O JEDNOM SVOJSTVU EKVIVALENTNOM SA JEDINSTVENOŠĆU REŠENJA ČEBIŠEVLJEVOG ZADATKA NAJBOLJE APROKSIMACIJE U NORMIRANIM LINEARNIM REALNIM VEKTORSKIM PROSTORIMA

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Pokazano je da je osobina jedinstvenosti rešenja ČEBIŠEVljevog zadatka najbolje aproksimacije u prostoru iz V (klasa svih normiranih linearnih realnih vektorskih prostora) ekvivalentna sa izvesnom geometrijskom osobinom. Na osnovu toga uveden je pojam projekcije i klasa odgovarajućih prostora W ($W\subset V$) koja, na izvestan način, zadržava neke osobine ortogonalne projekcije u HILBERTovim prostorima (jedinstvenost projekcije na ravan i paralelnost duži koje spajaju originale sa projekcijama). Pri tome je sa dva primera pokazano da se klasa *W ne poklapa* — ni sa klasom V ni sa klasom pred-HILBERTovih ni sa klasom BANACHovih prostora.