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742. ON SOME FUNCTIONAL INEQUALITIES I*

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In this and in a few forthcoming short notes, generally speaking, we shall deal with some classes of functional inequalities whose exact solutions can be reduced to a corresponding differential equations. The cheaf aim of our notes is to show that the considered functional inequalities are of the sort, that under the certain assumptions, can be reduced to such differential equation whose exact solution can be founded in a closed form. By using some appropriate method in this way we shall, in all the cases, obtain the neccessary form of the solution of the considered functional inequality. Hence, the solution of the considered functional inequality is neccessary contained in the set of all the solutions of the corresponding differential equation. In some of the cases, by using this method we can also obtain the exact solution of the given functional inequality. By the way, let us underline here that in this series of notes we shall consider the functional inequalities whose form immediately implies the differentiability of the solution of the same inequality which is not the usual property of the functional inequalities.

1. The cheaf aim of this note is contained in a result obtained in the paper [**1**]. Namely, in the mentioned paper the following theorem is proved:

Theorem A. *Consider the functional inequality*

(1) $$f(x+y) \geqq H(x, y)\, f(x)\, g(y) + G(x, y),$$

where given functions $x \mapsto g(x)$, $(x, y) \mapsto H(x, y)$ *and* $(x, y) \mapsto G(x, y)$ *satisfy the conditions:*

1° $g(x)$ *is differentiable at* $x=0$; $H(x, y)$ *and* $G(x, y)$ *have the partial derivatives* $\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, \frac{\partial G}{\partial x}$ *and* $\frac{\partial G}{\partial y}$ *on the x-axis*;

2° $g(0)\, H(x, 0)=1,\ G(x, 0)=\frac{\partial G(x, 0)}{\partial x}=\frac{\partial H(x, 0)}{\partial x}=0.$

Then the most general solution of (2) *is neccessarily of the form*

(2) $$f(x)=e^{\int p(x)\,dx}\Big(C+\int q(x)\, e^{-\int p(x)\,dx}\, dx\Big),$$

where $p(x)=g'(0)\, H(x, 0)+g(0)\frac{\partial H(x, 0)}{\partial y}$, $q(x)=\frac{\partial G(x, 0)}{\partial y}$ *and C is a constant.*

In the mentioned paper [**1**], beside the above theorem and it's proof there are some examples which are used to show the possibilities of the same theorem. Let us observe immediately that it is possible to consider the following functional inequality

(3) $$f(x+y) \geqq F(x, y)\, f(x)+G(x, y)$$

instead of (1), which can be reduced to (1) if $F(x, y)=H(x, y)\, g(y)$.

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Starting from such simple statement we can directly prove the following theorem:

Theorem 1. *Let us suppose that the functional inequality* (3) *is given, where the functions* $(x, y) \mapsto F(x, y)$, $(x, y) \mapsto G(x, y)$ *satisfy the following conditions*:

1° *The partial derivatives* $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$ *exist at all points of x-axis*;

2° $F(x, 0)=1,\ G(x, 0)=0$; 3° $\frac{\partial G(x, 0)}{\partial x}=\frac{\partial F(x, 0)}{\partial x}=0.$

Then, every solution of the functional inequality (3) *is neccessary of the following form*:

$$f(x)=e^{\int p(x)\,dx}\left(C+\int q(x)\, e^{-\int p(x)\,dx}\, dx\right), \tag{4}$$

where $p(x)=\frac{\partial F(x, 0)}{\partial y}$, $q(x)=\frac{\partial G(x, 0)}{\partial y}$ *and C is an arbitrary constant.*

In this way we have obtained a simple generalization of theorem *A*. Since the proof of theorem 1 is simple modification of the proof of theorem *A* from **[1]**, it will be omited. The only reason why the theorem 1 is formulated here is it's general and simple formulation and is used for better understanding our following results. In this connection let us observe only that theorem *A* can be obtained from that of 1, but the reverse conclusion is not possible in the general case.

2. In this part of the present paper we shall give a generalization of the above cited theorem *A*. As it can be seen from the paper **[1]**, the solution of the functional inequality (1) i. e. (3) can be reduced to the solution of the corresponding linear differential equation of the first order. In this paper we shall consider a functional inequality whose form is such that it's solution can be reduced to the corresponding BERNOULLI's first order differential equation. Namely, we shall prove the following theorem:

Theorem 2. *Let us suppose that the real continuous function f satisfies the following inequality*

$$f(x+y) \geqq H(x, y)\, f(x)+G(x, y)\, f^n(x)^{*)} \tag{5}$$

where $n=0, 1, 2, \ldots$, *and where the given functions* $(x, y) \mapsto H(x, y)$, $(x, y) \mapsto G(x, y)$ *satisfy the following conditions*:

1° *All of the partial derivatives* $\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$ *exist and are finite at all points of the form* $(x, 0)$, $x \in \mathbf{R}$;

2° $H(x, 0)=1$, $G(x, 0)=0$ *for all* $x \in \mathbf{R}$;

3° $\frac{\partial H(x, 0)}{\partial x}=\frac{\partial G(x, 0)}{\partial x}=0.$

Then: (a) *the function f is differentiable*,

(b) *the same function f satisfies the following differential equation*

$$f'(x)=p(x)\, f(x)+q(x)\, f^n(x)$$

*) — See the remark e) on the end of this paper.

where we have

$$p(x)=\frac{\partial H(x,0)}{\partial y},\quad q(x)=\frac{\partial G(x,0)}{\partial y}.$$

Proof. Let us suppose that $h>0$ is given. If we substitute $y=h$ in (5) then we obtain $f(x+h)\geqq H(x,h)\,f(x)+G(x,h)\,f^n(x)$ wherefrom by evident transformations we get the following inequality

$$\text{(6)}\qquad \frac{f(x+h)-f(x)}{h}\geqq\frac{H(x,h)-1}{h}f(x)+\frac{G(x,h)}{h}f^n(x).$$

If instead of x and y we introduce $x+h$ and $-h$ in (5), then we have

$$\text{(7)}\qquad f(x)\geqq H(x+h,\,-h)\,f(x+h)+G(x+h,\,-h)\,f^n(x+h).$$

Let us suppose that function φ is defined by $\varphi(t)=H(x+t,\,-t)$, $t\in\mathbf{R}$. On the basis of the assumption 2° it follows that $\varphi(0)=1$. On the other side, from the assumption 1° the above function φ is differentiable and therefore is continuous at the point $t=0$. Hence, from the above reasons for suitable chosen small $h>0$, we have $H(x+h,\,-h)>0$. Using that fact and from (7) directly follows that we have

$$\text{(8)}\qquad \frac{f(x+h)-f(x)}{h}\leqq f(x)\frac{1-H(x+h,\,-h)}{h\,H(x+h,\,-h)}-f^n(x+h)\frac{G(x+h,\,-h)}{h\,H(x+h,\,-h)}.$$

In such a way, for every, but suitable chosen small $h>0$ and on the basis of (6) and (8) it follows that following inequalities

$$\text{(9)}\qquad \begin{aligned}&\frac{H(x,h)-1}{h}f(x)+\frac{G(x,h)}{h}f^n(x)\leqq\frac{f(x+h)-f(x)}{h}\leqq\\ &\leqq f(x)\frac{1-H(x+h,\,-h)}{h\,H(x+h,\,-h)}-f^n(x+h)\frac{G(x+h,\,-h)}{h\,H(x+h,\,-h)}\end{aligned}$$

are valid. It can be directly verified that in virtue of all the suppositions of our theorem we have

$$\text{(10)}\qquad \lim_{h\to0+}\left(\frac{H(x,h)-1}{h}f(x)+\frac{G(x,h)}{h}f^n(x)\right)=\frac{\partial H(x,0)}{\partial y}f(x)+\frac{\partial G(x,0)}{\partial y}f^n(x).$$

Let us introduce the following function $\psi(t)=G(x+t,\,-t)$, $t\in\mathbf{R}$. On the basis of that notation and the notations we have introduced above, as on the basis of the suppositions of our theorem, we have:

$$\begin{aligned}\text{(11)}\qquad &\lim_{h\to0+}\left(f(x)\frac{1-H(x+h,\,-h)}{h\,H(x+h,\,-h)}-f^n(x+h)\frac{G(x+h,\,-h)}{h\,H(x+h,\,-h)}\right)=\\ &=-f(x)\cdot\lim_{h\to0+}\frac{\varphi(h)-\varphi(0)}{h\cdot\varphi(h)}-\lim_{h\to0+}f^n(x+h)\cdot\lim_{h\to0+}\frac{\psi(h)-\psi(0)}{h\cdot\varphi(h)}\\ &=-f(x)\frac{\varphi'(0)}{\varphi(0)}-f^n(x)\frac{\psi'(0)}{\varphi(0)}=-f(x)\left(\frac{\partial H(x,0)}{\partial x}-\frac{\partial H(x,0)}{\partial y}\right)-\\ &-f^n(x)\left(\frac{\partial G(x,0)}{\partial x}-\frac{\partial G(x,0)}{\partial y}\right)=f(x)\frac{\partial H(x,0)}{\partial y}+f^n(x)\frac{\partial G(x,0)}{\partial y}.\end{aligned}$$

In virtue of (10) and (11), if we let $h \to 0_+$ in (9) we can conclude that there exists the right hand derivative f_+' of the function f and that we have

$$(12) \qquad f_+'(x) = p(x)\, f(x) + q(x)\, f^n(x).$$

In the similar way it can be proved that there exists the left hand derivative f_-' of the function f and that we have

$$(13) \qquad f_-'(x) = p(x)\, f(x) + q(x)\, f^n(x).$$

Hence, by using (12) and (13) we can conclude that the statements (a) and (b) of our theorem holds, which proves the same theorem.

The preceding result for $n=0$ and $n=1$ is contained in theorem 1. This is a reason why we shall suppose in the further text that n is natural number not less then 2. Our following theorem is of the formal character and refers to such values of n and those values of the real variable x for which all the functions we consider are defined.

Theorem 3. *Let us suppose that the continuous function f and the functions $(x, y) \mapsto H(x, y)$ and $(x, y) \mapsto G(x, y)$ satisfy all the suppositions of theorem* 2. *Then every solution of the functional inequality* (5), *where we have $n \geqq 2$ (n is natural number), neccessarily has the following form*

$$(14) \qquad f^{1-n}(x) = e^{(1-n)\int p(x)\,dx}\left(C + (1-n)\int q(x)\, e^{(n-1)\int p(x)\,dx}\, dx\right),$$

where C is an arbitrary real constant and where the functions p and q are defined as in theorem 2.

3. In this part of our note we shall give a few remarks which are related to the preceding results:

a) For $n=0$ our theorem 3 is reduced to theorem A i.e. theorem 1. Hence, the theorem 3 is a generalization of these theorems. This follows immediately from the fact that the BERNOULLI's equation is in some sense more general then the linear differential equation of the first order.

b) It seems to us that it is of interest to show that in some particular cases it is possible to reduce the inequality (5) to the inequality (3).

c) As it is shown above, every solution of the inequality (5) necessarily has the form (14). But, every function of the form (14) does not has to satisfy (5) for every C, x and y. In another words, by using the aforementioned method it is not possible to obtain the solution of the inequality (5) in every i. e. general case. On the other side, it is possible to show in a simple way that in some cases our method is effective i. e. we can obtain the exact solution of the considered inequality. This was shown also in paper **[1]**.

d) The form of the solution of the inequality (5) can be obtained in a different way and different method where it is possible to use some weakened assumptions. Those methods will be considered in one of the following papers of the same author.

e) It is evident that the function f given by $f(x)=0$ for every $x \in \mathbf{R}$ satisfies the functional inequality (5). Meanwhile, our preceding results are refered to the nontrivial solutions of the functional inequality (5) i. e. corresponding BERNOULLI's equation.

4. The method which was used in the proof of our theorem 2 can be applied in considering various classes of functional inequalities. In this, last part of the paper, we shall give a few theorems which contain some classes of functional inequalities which can be solved in that way. The proofs of those theorems will be omited and published elsewhere.

Theorem 4. *Let us consider the following functional-differential inequality*

$$f'(x+y) \geqq H(x, y)\, f'(x) + G(x, y)\, f(x) + F(x, y), \tag{15}$$

where the functions $(x, y) \mapsto H(x, y)$, $(x, y) \mapsto G(x, y)$ *and* $(x, y) \mapsto F(x, y)$ *are given and where we suppose that:*

1° *The functions* H, G *and* F *possess the partial derivatives* $\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ *at all points of the form* $(x, 0)$ *for every* $x \in \mathbf{R}$;

2° $H(x, 0) = 1$, $G(x, 0) = F(x, 0) = 0$ *for all* $x \in \mathbf{R}$;

3° $\frac{\partial H(x, 0)}{\partial x} = \frac{\partial G(x, 0)}{\partial x} = \frac{\partial F(x, 0)}{\partial x} = 0.$

Then (a) *There exist the second derivative of the function* f;

(b) *Every solution of the functional-differential inequality* (15) *necessarily satisfies the equation*

$$f''(x) = p(x)\, f'(x) + q(x)\, f(x) + r(x),$$

where the functions p, q *and* r *are defined respectively by*

$$p(x) = \frac{\partial H(x, 0)}{\partial y}, \quad q(x) = \frac{\partial G(x, 0)}{\partial y} \quad \text{and} \quad r(x) = \frac{\partial F(x, 0)}{\partial y}.$$

Theorem 5. *Let us consider the functional inequality*

$$f(x+y) \geqq A(x, y)\, f(x) + B(x, y)\, f^2(x) + C(x, y), \tag{16}$$

where we suppose that the given functions $(x, y) \mapsto A(x, y)$, $(x, y) \mapsto B(x, y)$ *and* $(x, y) \mapsto C(x, y)$ *satisfy the following conditions:*

1° *All of the partial derivatives* $\frac{\partial A}{\partial x}, \frac{\partial A}{\partial y}, \frac{\partial B}{\partial x}, \frac{\partial B}{\partial y}, \frac{\partial C}{\partial x}, \frac{\partial C}{\partial y}$ *exist at all points of the form* $(x, 0)$ *for all* $x \in \mathbf{R}$;

2° $A(x, 0) = 1$, $B(x, 0) = 0$, $C(x, 0) = 0$;

3° $\frac{\partial A(x, 0)}{\partial x} = \frac{\partial B(x, 0)}{\partial x} = \frac{\partial C(x, 0)}{\partial x} = 0.$

Then for every real continuous function f *which satisfies* (16) *the following is true:*

(a) *The function* f *is differentiable;*

(b) *Every solution* f *of the functional inequality* (16) *necessarily satisfies the following differential equation*

$$f'(x) = p(x)\, f(x) + q(x)\, f^2(x) + r(x),$$

where the functions p, q and r are given by

$$p(x)=\frac{\partial A(x,0)}{\partial y},\quad q(x)=\frac{\partial B(x,0)}{\partial y}\quad \textit{and}\quad r(x)=\frac{\partial C(x,0)}{\partial y}.$$

Generally speaking the proofs of the above theorems 4 and 5 are similar to that of theorems 2 and 3. In the same time those proofs contains some differences which are implied by different forms of the considered inequations.

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O NEKIM FUNKCIONALNIM NEJEDNAKOSTIMA

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U ovom se radu razmatraju neke klase funkcionalnih nejednakosti iz čijeg oblika proizilazi diferencijabilnost rešenja. Dokazuje se da se one svode na takve diferencijalne jednačine čije se rešenje može naći u zatvorenoj formi.