UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 735—№ 762 (1982), 32—34

740. CONTINUOUS LINEAR OPERATORS DEFINED ON THE CONE OF FUNCTIONS CONVEX WITH RESPECT TO CHEBYSHEV SYSTEM*

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0. In this article, the necessary and sufficient conditions for linear continuous operator A defined on the space $C[a, b]$, in order that the implication (4) holds, are given. In such a way, this result extends, in the case of Chebyshev system with two terms, some theorems of S. KARLIN and A. NOVIKOFF [1], J. TZIMBALARIO [2] and P. M. VASIĆ and I. B. LACKOVIĆ [3].

1. Let $C[a, b]$ denotes the space of all continuous real functions defined on $[a, b]$.

Definition 1. *We say that a function $f \in C[a, b]$ is convex with respect to Chebyshev system $\{u_0, u_1\}$ if* (see [4])

$$(1)\qquad \begin{vmatrix} u_0(x_1) & u_1(x_1) & f(x_1) \\ u_0(x_2) & u_1(x_2) & f(x_2) \\ u_0(x_3) & u_1(x_3) & f(x_3) \end{vmatrix} \geqq 0$$

for every $a < x_1 < x_2 < x_3 < b$. The cone of all f satisfying (1) *will be denoted by $C(u_0, u_1)$.*

In [6], the following theorems are proved:

Theorem 1. *Every function from the sequence*

$$(2)\qquad \varphi_n(x) = A_0 u_0(x) + B_0 u_1(x) + \sum_{k=1}^{n-1} m_k \varphi(x; x_k),$$

where A_0, B_0 are arbitrary real constants, $m_k \geqq 0$ $(k = 1, 2, \ldots, n-1)$, $a < x_1 < x_2 < \cdots < x_{n-1} < b$,

$$(3)\qquad \varphi(x; c) = \begin{cases} 0, & a \leqq x \leqq c, \\ w_0(x) \int\limits_c^x w_1(t)\, dt, & c \leqq x \leqq b, \end{cases}$$

where w_0 and w_1 are positive functions and w_0', $w_1 \in C[a, b]$, belongs to the cone $C(u_0, u_1)$.

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* Presented by I. B. LACKOVIĆ.

Ovaj rad je finansirala Republička Zajednica Nauke Srbije.

Theorem 2. *Every function $f \in C[a, b]$ continuous from the right in $x = a$ and from the left in $x = b$ is an uniform limit of the sequence of generalized polygonal lines $(\varphi_n)_0^\infty$ defined by* (3), *where A_0, B_0, m_k and $\varphi(x; c)$ have the same meaning as in the theorem* 1.

2. Let the space $C[a, b]$ be normed by the usual norm $f = \max_{a \leqq t \leqq b} |f(t)|$, and let $S(E)$ be a normed subspace of all real functions defined on E, with the norm $\|f\|_E$. Further, let $A: C[a, b] \to S(E)$ be a linear, continuous operator. In other words, for every $f, g \in C[a, b]$ and arbitrary $\lambda \in R$, we have $A(f + \lambda g) = Af + \lambda Ag$, and for every $f \in C[a, b]$ for which exists a sequence $(f_n)_0^\infty$ so that $\|f_n - f\| \to 0$ when $n \to \infty$, it always holds $\|Af_n - Af\|_E \to 0$.

Now, the following theorem is valid.

Theorem 3. *Let the linear and continuous operator $A: C[a, b] \to S(E)$ be given. Then, for every function f, the implication*

$$f \in C(u_0, u_1) \Rightarrow Af \geqq 0 \tag{4}$$

is valid if and only if A fulfils the following conditions:

(5) $Au_0 = 0$, (6) $Au_1 = 0$, (7) $A\varphi(x; c) \geqq 0$ *for every* $c \in [a, b]$,

where $\varphi(x; c)$ is given by (3).

Proof. Suppose that for every f implication (4) holds. Then, as $\pm u_0$, $\pm u_1 \in C(u_0, u_1)$ we have $\pm Au_0 \geqq 0$ and $\pm Au_1 \geqq 0$, in virtue of homogenity of A, so (5) and (6) are valid. It is proved in [5] that the function $x \mapsto \varphi(x; c)$, given by (3), for every $c \in [a, b]$ belongs to the cone $C(u_0, u_1)$, thus, on the basis of (4), (7) holds true.

Let's prove the sufficience of the conditions (5)—(7). Namely, for every $f \in C(u_0, u_1)$, there exist, in virtue of the theorem 2, the constants A_0 and B_0, so and the sequence $m_k \geqq 0$ $(k = 1, 2, \ldots, n-1)$ so that

$$\lim_{n \to \infty} \varphi_n = f. \tag{8}$$

On the basis of continuity of A and (8)

$$\lim_{n \to \infty} \|A\varphi_n - Af\|_E = 0 \tag{9}$$

holds.

On the other hand, if we apply the operator A on both sides of (2), in virtue of his linearity, we get

$$A\varphi_n = A_0 Au_0 + B_0 Au_1 + \sum_{k=1}^{n-1} m_k A\varphi(x; x_k),$$

which, using (5), (6) and (7), takes the form

$$A\varphi_n = \sum_{k=1}^{n-1} m_k A\varphi(x; x_k).$$

The constants m_k $(k=1, 2, \ldots, n-1)$ are, by the supposition, nonnegative, so from (9) we have $Af=A(\lim \varphi_n)=\lim A\varphi_n \geqq 0$, i.e. (4) holds.

REMARK 1. The set $W=C(u, u_1)\cap\{-C(u_0, u_1)\}$ is the maximal vector subspace contains in the convex cone $C(u_0, u_1)$, i.e. (see [6]) for every $\alpha, \beta \in \mathbf{R}$, $\alpha u_0+\beta u_1 \in W$. Then, the conditions (5) and (6) can be rewritten in the form

$$AW=0. \tag{10}$$

In other words, (10) means that W is a subset of kernel of operator A.

REMARK 2. S. KARLIN and A. NOVIKOFF, the authors of [1], consider also the implication (4), but indeed an arbitrary linear continuous operator, they restrict themself only on the functionals of the form $\int_a^b f\,d\mu$. However, their results are more general in the other sense. Namely, they permit the convexity with respect to the system $\{u_0, u_1, \ldots, u_n\}$ where n is an arbitrary natural number.

REMARK 3. J. TZIMBALARIO in [2] find out the necessary and sufficient conditions that the implication

$$f\in C(u_0, u_1, \ldots, u_n) \Rightarrow Af\in C(u_0, u_1, \ldots, u_n) \tag{11}$$

holds, for every $f\in C[a, b]$, where A is an arbitrary continuous operator. It is clear that, for $n=1$, our implication (4) contains (11).

REMARK 4. The statement of theorem 3 extends the results obtain in [2] and [5]. Thus, for $u_0(x)=1$, $u_1(x)=x$ we have the theorem 2 in [2] and for $u_0(x)=\sin rx$ (or $\operatorname{sh} rx$), $u_1(x)=\cos rx$ (or $\operatorname{ch} rx$) our theorem 3 reduces on the result of [5].

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NEPREKIDNI LINEARNI OPERATORI DEFINISANI NA KONUSU FUNKCIJA KONVEKSNIH U ODNOSU NA ČEBIŠEVLJEV SISTEM

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U radu su dati potrebni i dovoljni uslovi za linearni operator A definisan na prostoru $C[a, b]$, tako da važi implikacija (4), gde je $C(u_0, u_1)$ konus funkcije konveksnih u odnosu na ČEBIŠEVljev sistem $\{u_0, u_1\}$.