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737. NOTE ON MULTIDIMENSIONAL GENERALIZATIONS OF ČEBYŠEV'S INEQUALITY*

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The following two multidimensional generalizations of the well-known ČEBYŠEV inequality were proved by L. VIETORIS [5] in 1974.

Theorem A. *If* $a_{i_1 \dots i_r}$ *and* $b_{i_1 \dots i_r}$ $(i_k = 1, \dots, m_k,\ k = 1, \dots, r)$ *are real-valued functions of indices* $i_1, \dots, i_r$ *then*

$$\prod_k m_k \sum_i a_i b_i \geqq \sum_i a_i \sum_i b_i \tag{1}$$

holds, where the summation is over all combinations $i = (i_1, \dots, i_r)$ *of indices, provided that for every two combinations* i *and* j *for which* $i_k \leqq j_k$ $(k = 1, \dots, r)$, *both* $a_i \leqq a_j$ *and* $b_i \leqq b_j$.

Theorem B. *Let* $f(x)$ *and* $g(x)$ *be two nondecreasing functions on*

$$X = \{x = (x_1, \dots, x_r) \mid a_k \leqq x_k \leqq b_k,\ 1 \leqq k \leqq r\}.$$

Then

$$\prod_k (b_k - a_k) \int_X fg \, dX \geqq \int_X f dX \cdot \int_X g \, dX. \tag{2}$$

It was already known that (1) is also valid if (a_i) and (b_i) are *similarly ordered* (cf. [5]), and similarly that (2) is valid if f and g are similarly ordered. (See Definition 2 below). In this paper we shall prove two generalizations of these results which will include both kinds of hypotheses on the functions, and include both sums and integrals. First we require some definitions. Here, and in all that follows, X is as in Theorem B.

Definition 1. *For* $m \geqq 2$, *functions* $f_j : X \to \mathbf{R}$ $(j = 1, \dots, m)$ *are monotonic in the same sense if either each* $f_j(x_1, \dots, x_r)$ *is nondecreasing in each* x_k $(1 \leqq k \leqq r)$ *for arbitrary values of* $x_i \in [a_i, b_i]$ $(i \neq k)$, *or each* f_j *is nonincreasing in each* x_k *for arbitrary values of the other* x_i.

It is easy to see that this is the same as: *either*

$$f_j(x) \leqq f_j(y) \text{ for } 1 \leqq j \leqq m \text{ when } a_k \leqq x_k \leqq y_k \leqq b_k \qquad (1 \leqq k \leqq r),$$

or

$$f_j(x) \geqq f_j(y) \text{ for } 1 \leqq j \leqq m \text{ when } a_k \leqq x_k \leqq y_k \leqq b_k \qquad (1 \leqq k \leqq r).$$

In the case $m = 2$, functions $f_1, f_2 : X \to \mathbf{R}$ *are monotonic in the opposite sense* if $f_1, -f_2$ are monotonic in the same sense.

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Definition 2. *For $m \geqq 2$, functions $f_k: X \to \mathbf{R}$ $(1 \leqq k \leqq m)$ are similarly ordered if*

$$(f_i(x)-f_i(y))\,(f_j(x)-f_1(y)) \geqq 0 \text{ for } 1 \leqq i,\ j \leqq m, \text{ all } x, y \in X.$$

In the case $m=2$, the functions f_1, f_2 are oppositely ordered if f_1, $-f_2$ are similarly ordered.

In the one-dimensional case ($r=1$) it is clear that if the functions (f_k) are monotonic in the same sense then they are similarly ordered, but not conversely. However for $r \geqq 2$ neither conditon implies the other. For example, if all $f_k(x) \equiv g(x)$ $(1 \leqq k \leqq m)$ then the functions (f_k) are clearly similarly ordered for *any* g, even one which is not monotonic in any of its variables. On the other hand, for $r \geqq 2$ the $m=r$ functions f_k defined by $f_k(x) \equiv x_k$ $(1 \leqq k \leqq r)$ are clearly all monotonic in the same sense (increasing), but are not similarly ordered since

$$(f_i(x)-f_i(y))\,(f_j(x)-f_j(y)) = (x_i-y_i)\,(x_j-y_j) < 0 \qquad (1 \leqq i \neq j \leqq r)$$

if $x_i < y_i$ but $x_j > y_j$ for some i, j.

Theorem 1. *Let $f, g: X \to \mathbf{R}$ be two continuous functions which are either similarly ordered or monotonic in the same sense, and let $u_k: [a_k, b_k] \to \mathbf{R}$ be nondecreasing functions, $1 \leqq k \leqq r$. Then*

$$\int_X du(x) \cdot \int_X f(x)\,g(x)\,du(x) \geqq \int_X f(x)\,du(x) \cdot \int_X g(x)\,du(x). \tag{3}$$

where $du(x) = du_1(x_1) \cdots du_r(x_r)$. If f, g are oppositely ordered, or monotonic in the opposite sense, then the opposite inequality to (3) *holds.*

Proof. It f, g are similarly ordered, then

$$(f(x)-f(y))\,(g(x)-g(y)) \geqq 0, \text{ all } x, y \in X,$$

so

$$\int_X \int_X (f(x)\,g(x) - f(x)\,g(y) - f(y)\,g(x) + f(y)\,g(y))\,du(x)\,du(y) \geqq 0$$

which reduces to (3) If f, g are monotonic in the same sense, we proceed by induction on r. For $r=1$, f and g are also similarly ordered and so (3) holds by what was just proved. Suppose that (3) holds true for some $r \geqq 1$, and for $r+1$, write $(x, s) = (x_1, \dots, x_r, s)$. Then

$$\int_X \int_{a_{r+1}}^{b_{r+1}} du(x)\,du_{r+1}(s) \cdot \int_X \int_{a_{r+1}}^{b_{r+1}} f(x,s)\,g(x,s)\,du(x)\,du_{r+1}(s)$$

$$= \int_{a_{r+1}}^{b_{r+1}} du_{r+1}(s) \cdot \int_{a_{r+1}}^{b_{r+1}} \Big(\int_X du(x) \cdot \int_X f(x,t)\,g(x,t)\,du(x)\Big)\,du_{r+1}(t)$$

$$\geqq \int_{a_{r+1}}^{b_{r+1}} du_{r+1}(t) \int_{a_{r+1}}^{b_{r+1}} \Big(\int_X f(x,t)\,du(x)\Big)\Big(\int g(x,t)\,du(x)\Big)\,du_{r+1}(t)$$

$$\geqq \int_X \int_{a_{r+1}}^{b_{r+1}} f(x,t)\,du(x)\,du_{r+1}(t) \cdot \int_X \int_{a_{r+1}}^{b_{r+1}} g(x,t)\,du(x)\,du_{r+1}(t),$$

because the functions $x \mapsto f(x, t)$ and $x \mapsto g(x, t)$ are monotonic in the same sense for each $t \in [a_{r+1}, b_{r+1}]$ at the first inequality sign, and the functons $t \mapsto \int\limits_X f(x, t)\,\mathrm{d}u(x)$, $t \mapsto \int\limits_X g(x, t)\,\mathrm{d}u(x)$ are both monotonic in the same sense at the last inequality sign. Hence (3) also holds for all $r \geqq 1$ when f, g are monotonic in the same sense.

When f, g are either oppositelly ordered or are monotonic in the opposite sense, the reverse inequality follows by applying (3) to the functions $f, -g$.

The inequality (3) when f, g are similarly ordered and all $u_k(t) \equiv t$ is given in HARDY, LITTLEWOOD and PÓLYA [3; p. 168]. In a paper published in 1967 [1], an inequality of the general form (3) appeared for the case that f, g are similarly ordered; it was also claimed that this condition was not only sufficient but was also necessary for the validity of (3) on every subregion $X_1 \subset X$. For a discussion of this (false) assertion, see [2].

We now use Theorem 1 to prove

Theorem 2. *Let the functions $f_1, \ldots, f_m : X \to \mathbf{R}$ be continuous, nonnegative, and either similarly ordered or monotonic in the same sense. If $u_k : [a_k, b_k] \to \mathbf{R}$ are nondecreasing functions, then*

$$(4) \qquad \Bigl(\int\limits_X \mathrm{d}u(x)\Bigr)^{m-1} \int\limits_X f_1(x)\cdots f_m(x)\,\mathrm{d}u(x) \geqq \prod_{j=1}^{m} \int\limits_X f_j(x)\,\mathrm{d}u(x),$$

where X and du *are as in Theorem* 1.

Proof. Under either hypothesis, we may proceed by induction on m. For $m=1$, (4) is trivially true and for $m=2$, (4) reduces to (3). Suppose that (4) holds for some $m \geqq 2$, and that all f_j $(1 \leqq j \leqq m+1)$ are nonnegative, continuous, and either similarly ordered, or monotonic in the same sense. Set $F(x) = \prod\limits_1^m f_j(x)$, $G(x) = f_{m+1}(x)$. If all f_j are monotonic in the same sense, so are F, G by Definition 1, because all $f_j \geqq 0$. If all f_j are similarly ordered, so are F, G that is

$$(5) \qquad (F(x)-F(y))\,(G(x)-G(y)) \geqq 0, \quad \textit{all } x, y, \in X.$$

The inequality (5) clearly holds if $G(x) = G(y)$. For other $x, y \in X$ we have either $G(x) = f_{m+1}(x) < f_{m+1}(y) = G(y)$, or $G(x) > G(y)$. In the first case it follows from Definition 2 that we must also have $f_i(x) \leqq f_i(y)$ for $1 \leqq i \leqq m$, whence $F(x) \geqq F(y)$ since all $f_i \geqq 0$. Similarly if $G(x) > G(y)$, then $F(x) \geqq F(y)$ must hold, and so (5) holds for all $x, y \in X$.

But now, by the case $m=2$ and the inductions assumption we have

$$\Bigl(\int\limits_X \mathrm{d}u\Bigr)^m \int\limits_X \prod_1^{m+1} f_j(x)\,\mathrm{d}u = \Bigl(\int\limits_X \mathrm{d}u\Bigr)^{m-1} \Bigl(\int\limits_X \mathrm{d}u\Bigr) \int\limits_X F(x)\,G(x)\,\mathrm{d}u$$

$$\geqq \Bigl(\int\limits_X \mathrm{d}u\Bigr)^{m-1} \int\limits_X \prod_1^m f_j(x)\,\mathrm{d}u \cdot \int\limits_X f_{m+1}(x)\,\mathrm{d}u \;\geqq \prod_1^{m+1} \int\limits_X f_i(x)\,\mathrm{d}u.$$

Special cases of inequalities of the form (3) involving a product of more that two functions were proved as long ago as 1883 by C. ANDRÉIEF. See [4; (9.2)] for a discussion of this and many other historical references to the ČEBYŠEV inequality.

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NOTA O MULTIDIMENZIONALNOJ GENERALIZACIJI ČEBYŠEVLJEVE NEJEDNAKOSTI

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U radu je data multidimenzionalna generalizacija ČEBYŠEVljeve nejednakosti za monoone funkcije, kako za dve tako i za *n* funkcija. Dobijeni rezultati su uopštenje VIETORISovih rezultata za ČEBYŠEVljevu nejednakost.