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736. ON A GENERALISATION OF ESSEEN'S INEQUALITY*

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C. G. ESSEEN has proved (see [1])

$$F(-x-y)\leqq 2\,F(-x)\,F(-y)\quad (x,\,y\geqq 0), \tag{1}$$

where $F(x)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{x} e^{-t^2/2}\,\mathrm{d}t.$

A generalisation of the inequality (1) is given in the present paper under weaker conditions for subintegral function.

Theorem 1. *Let the function F be defined by*

$$F(x)=c\int_{-\infty}^{x} e^{-h(t)}\,\mathrm{d}t\quad (c>0), \tag{2}$$

where h is convex differentiable function on $(-\infty,\,+\infty)$ *and* $\lim\limits_{x\to-\infty} h(x)=+\infty$. *Then*

$$F(-x-y)\leqq \lambda\,F(-x)\,F(-y)\quad (x,\,y\geqq 0), \tag{3}$$

where $\lambda=\dfrac{1}{F(0)}$. *The equality does hold if and only if x or y is the limit point of the closed interval* $[-\infty,\,0]$.

Proof. First, we shall prove that $F(0)$ is finite, i.e. the integral $\int_{-\infty}^{0} e^{-h(t)}\,\mathrm{d}t$ is convergent. Assume that the function h has the stationary point $x_0\in(-\infty,\,0)$. In virtue of the condition $h(-\infty)=+\infty$ it follows that h is decreasing on the interval $(-\infty,\,x_0)$. Let us put $x_1=x_0-a$, where $a\in R^+$. If $x_0>0$ or h has not a stationary point (i.e. h is monotonic decreasing on $(-\infty,\,+\infty)$), we shall take $x_1=0$. Since the integral $\int_{x_1}^{0} e^{-h(t)}\,\mathrm{d}t$ is finite, the integral $\int_{-\infty}^{0} e^{-h(t)}\,\mathrm{d}t$ will be convergent if and only if $\int_{-\infty}^{x_1} e^{-h(t)}\,\mathrm{d}t<+\infty.$

The tangent of the curve of the function h in x_1 has the form

$$y=h'(x_1)\,x+h(x_1)-x_1\,h'(x_1)=h'(x_1)\,x+b,$$

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where $b = h(x_1) - x_1 h'(x_1)$ and $h'(x_1) < 0$. On the interval $(-\infty, x_1)$ the curve of h is above of this tangent so that the inequality $h(x) > xh'(x_1) + b$ $(x < x_1)$ holds, wherefrom $e^{-h(x)} < e^{-b} e^{-h'(x_1)x}$ $(x < x_1)$.

Integrating the last relation we obtain

$$\int_{-\infty}^{x_1} e^{-h(x)}\, dx < e^{-b} \int_{-\infty}^{x_1} e^{-h'(x_1)x}\, dx = e^{-b} \left[\frac{e^{-h'(x_1)x}}{-h'(x_1)} \right]_{-\infty}^{x_1}$$

$$= \frac{-e^{-b} e^{-x_1 h'(x_1)}}{h'(x_1)} = -\frac{e^{-h(x_1)}}{h'(x_1)} < +\infty.$$

Therefore, the integral $\int_{-\infty}^{0} e^{-h(t)}\, dt$ is convergent so that $F(0)$ is finite. Since F is increasing function (because of $F'(x) = ce^{-h(x)} > 0$) it follows that F is limited for each $x \in (-\infty, 0)$.

Introduce the function P by $P(x, y) = \frac{F(-x-y)}{F(-x)}$ $(x, y \geqq 0)$.

By differentation we find

$$\frac{\partial P}{\partial x} = \frac{1}{F(-x)^2} \big(F'(x) F(-x-y) - F'(-x-y) F(-x)\big)$$

$$= \frac{(F-x)}{F(-x)^2} \left(F(-x-y) - \frac{F'(-x-y)}{F'(-x)} F(-x) \right)$$

$$= \frac{ce^{-h(-x)}}{F(-x)^2} \left(F(-x-y) - F(-x) e^{h(-x)-h(-x-y)} \right) = \frac{ce^{-h(-x)}}{F(-x)^2} Q(x, y),$$

where $Q(x, y) = F(-x-y) - F(-x) e^{h(-x)-h(-x-y)}$ $\quad (x, y \geqq 0)$.

Since h is convex, the derivative h' is increasing function so that $h'(-x) - h'(-x-y) \geqq 0$ for each $x \in (0, +\infty)$ and fixed $y \geqq 0$. From this

$$\frac{\partial Q(x, y)}{\partial x} = F(-x) \big(h'(-x) - h'(-x-y)\big) e^{h(-x)-h(-x-y)} \geqq 0.$$

Thus, Q is increasing function.

Since h is convex and $h(-\infty) = +\infty$, it follows that h is monotonic decreasing on the interval $(-\infty, x_0)$, where x_0 is the point of local minimum of the function h. Hence, for $x \geqq 0$ and $y \geqq 0$ the inequality $h(-x) - h(-x-y) < 0$ follows for $x > -x_0 \big(x_0 \in (-\infty, 0)\big)$, i.e. for each $x > 0$ if $x_0 > 0$ or the function h has not a stationary point. From above we conclude that

$$\lim_{x \to +\infty} e^{h(-x)-h(-x-y)} \in (0, 1).$$

In virtue of $F(-\infty) = 0$ we obtain $\lim\limits_{x \to +\infty} Q(x, y) = 0$. According to this we find $Q(x, y) \leqq \lim\limits_{x \to +\infty} Q(x, y) = 0$, so that $\frac{\partial P}{\partial x} \leqq 0$, i.e. the function P is monotonic decreasing on $(0, +\infty)$.

We have now $\frac{F(-x-y)}{F(-x)}=P(x, y)\leqq P(0, y)=\frac{F(-y)}{F(0)}=\lambda F(-y)$, which completes the proof of theorem. □

In particular case, if $c=(2\pi)^{-1/2}$ and $h(t)=\frac{t^2}{2}$ we obtain the inequality (1).

Let the function h, besides convexity, satisfies the condition $h(-x)=h(x)$. The following assertion is true.

Theorema. 2. *Let the function f be defined by*

$$f(x)=b\int_0^x e^{-h(kt)}\,dt \qquad (b>0,\ k\in\mathbf{R}). \tag{4}$$

Then

$$f(x)+f(y)-f(x+y)\leqq \mu\lambda f(x)f(y), \tag{5}$$

where $\mu=\frac{ck}{b}$ *and* $\lambda=\frac{1}{F(0)}$.

Proof. For $x\geqq 0$ we have

$$F(-x)=c\int_{-\infty}^{-x} e^{-h(t)}\,\mathrm{d}t=c\int_{-\infty}^{0} e^{-h(t)}\,\mathrm{d}t-c\int_{-x}^{0} e^{-h(t)}\,\mathrm{d}t.$$

Substituting $t=-ku$ in the second integral,, we obtain

$$F(-x)=F(0)-ck\int_0^{x/k} e^{-h(ku)}\,\mathrm{d}u=F(0)-\frac{ck}{b}f\left(\frac{x}{k}\right)$$

or

$$F(-kx)=F(0)-\mu f(x)\quad\left(\mu=\frac{ck}{b}\right). \tag{6}$$

Using (3) and (6) we find

$$F(0)-\mu f(x+y)\leqq\frac{1}{F(0)}\big(F(0)-\mu f(x)\big)\big(F(0)-\mu f(y)\big),$$

wherefrom the inequality (5) follows. □

In particular case, if $h(t)=\frac{t^2}{2}$, $k=\sqrt{2}$ and $b=\frac{2}{\sqrt{\pi}}$, we obtain the known inequality (see [3], [2, p. 285])

$$f(x)f(y)\geqq f(x)+f(y)-f(x+y).$$

The next theorem gives an inequality which can be connected with (5).

Theorem 3. *Let h be convex differentiable and nondecreasing function on $[0,+\infty)$ with the asymptote $s(x)=ax-b$ $(a>0,\ b>0)$. Then, for the function g, defined by*

$$g(x)=c\int_0^x e^{-h(t)}\,\mathrm{d}t \qquad (c>0,\ x\geqq 0),$$

the inequality

$$e^{-ay} g(x) \leqq g(x+y) - g(y) \qquad (y \geqq 0) \tag{7}$$

is valid. The equality holds if and only if x or y is the limit point of the closed interval $[0, +\infty]$.

Proof. On the basis of the mean value theorem we can write for $y \geqq 0$

$$h(x+y) - h(x) = yh'(x+\theta y) \leqq ay \qquad (0 \leqq \theta \leqq 1),$$

wherefrom $e^{-h(x)} \leqq e^{ay} \cdot e^{-h(x+y)}$.

After integration and multiplication with constant c, we obtain

$$c \cdot e^{-ay} \int_0^x e^{-h(t)} \, dt \leqq c \int_0^x e^{-h(t+y)} \, dt = c \int_y^{x+y} e^{-h(t)} \, dt = c \int_0^{x+y} e^{-h(t)} \, dt - c \int_0^y e^{-h(t)} \, dt,$$

i.e. $e^{-ay} g(x) \leqq g(x+y) - g(y) \qquad (x, y \geqq 0)$.

Substituting x and y one another, we get

$$e^{-ax} g(y) \leqq g(x+y) - g(x) \qquad (x, y \geqq 0). \tag{8}$$

Since both hand-side in (7) and (8) are positive, we can multiply the inequalities (7) and (8). We have

$$e^{-a(x+y)} g(x) g(y) \leqq g(x+y) \big(g(x+y) - g(x) - g(y)\big) + g(x) g(y),$$

i.e.

$$g(x) g(y) \geqq \frac{g(x+y)}{1 - e^{-a(x+y)}} \big(g(x) + g(y) - g(x+y)\big). \tag{9}$$

If the function h has not an asymptote, that is $a = \lim\limits_{x \to +\infty} \frac{h(x)}{x} = +\infty$, the inequality (9) becomes $g(x) g(y) \geqq g(x+y) \big(g(x) + g(y) - g(x+y)\big)$.

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O JEDNOM UOPŠTENJU ESSEENOVE NEJEDNAKOSTI

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U radu je dato jedno uopštenje Esseenove nejednakosti [1] pod slabijim uslovima za podintegralnu funkciju. Polazeći od predloženih rezultata dokazane su neke funkcionalne nejednakosti za konveksne funkcije.