UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 735 — № 762 (1982), 3—10.

735. ON GENERALIZED CONVEXITY PRESERVING MATRIX TRANSFORMATION*

Ljubiša M. Kocić

The necessary and sufficient condition for a real triangular matrix (p_{nk}) $(0\leq k\leq$ $\leq n,\ n=0,\ 1,\ldots)$ **have been investigated, so that the implication**

$$(a_{n+2}+(p+q)\,a_{n+1}+pq\,a_n\geq 0)\ \Rightarrow\ (s_{n+2}+(p+q)\,s_{n+1}+pq\,s_n\geq 0)$$

$(n=0,\ 1,\ldots)$ **holds, where is** $s_n=\sum_{k=0}^{n} p_{nn-k}\,a_k\quad (n=0,\ 1,\ldots).$

Two particular cases: 1° $p=q>0$ **and** 2° $p=1,\ q>0,\ q\neq 1$, **have been considered.**

1. Let $a=(a_n)\,(n\in\mathbf{N}_0)$ be an arbitrary real sequence. For real p and q we define difference operator $\Delta_{p,q}$ with

$$\Delta_{p,q}\,a_n=a_{n+2}-(p+q)\,a_{n+1}+pq\,a_n\quad (n\in\mathbf{N}_0). \tag{1.1}$$

It is easy to see that the following properties

$$\Delta_{p,q}\,(Ca_n)=C\,\Delta_{p,q}\,a_n\qquad (C\text{ — real constant}), \tag{1.2}$$

$$\Delta_{p,q}\,(a_n+b_n)=\Delta_{p,q}\,a_n+\Delta_{p,q}\,b_n, \tag{1.3}$$

hold.

Now, we shall introduce the concept of $(p,\ q)$-convexity of real sequence.

Definition 1. *We say that a sequence* $(a_n)\,(n\in\mathbf{N}_0)$ *is* $(p,\ q)$*-convex if*

$$\Delta_{p,q}\,a_n\geqq 0 \tag{1.4}$$

for every $n=0,\ 1,\ 2,\ldots$.

Note that, in a particular case $\Delta_{1,1}\equiv\Delta^2$, i. e. for $p=q=1$, $\Delta_{p,q}$ reduces on the well-known second difference and definition 1 becomes a definition of convex sequence of order 2.

Let $(p_{nk})\,(k=0,\ 1,\ \ldots,\ n;\ n=0,\ 1,\ 2,\ldots)$ be an infinite triangular matrix of real numbers, and let a sequence $(s_n)\,(n\in\mathbf{N}_0)$ be defined by

$$s_n=s_n\,(a)=\sum_{k=0}^{n} p_{nn-k}\,a_k\quad (n\in\mathbf{N}_0). \tag{1.5}$$

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* Presented by J. D. KEČKIĆ and J. L. REID.
Ovaj rad je finansirala Republička Zajednica Nauke Srbije.

It is clear that (1.5) represents a transformation T of a sequence a into a sequence s, i.e. $s=Ta$.

In [1], N. Ozeki gave the sufficient conditions for the triangular matrix (p_{nk}) so that for each convex sequence (a_n) the implication

$$(\Delta^2 a_n \geqq 0) \Rightarrow (\Delta^2 s_n(a) \geqq 0)\,(n \in \mathrm{N}_0), \tag{1.6}$$

is valid.

In [2], D. S. Mitrinović, I. B. Lacković and M. S. Stanković proved that these sufficient conditions are at the same time the necessary ones.

Putting $p_{n,k}^{(1)}=\sum_{j=0}^{k} p_{nj}$, $p_{n,k}^{(2)}=\sum_{j=0}^{k} p_{n,j}^{(1)}$, the complete theorem reads:

Theorem A. *The necessary and sufficient condition that the implication* (1.6) *is valid, for every sequence* (a_n), *where the sequence* (s_n) *is given by* (1.5), *is that the following conditions*

(a_1) $p_{n+2,n+2}^{(1)}-2p_{n+1,n+1}^{(1)}+p_{n,n}^{(1)}=0,$

(a_2) $p_{n+2,n+1}^{(2)}-2p_{n+1,n}^{(2)}+p_{n,n-1}^{(2)}=0,$

(a_3) $p_{n+2,n-k+2}^{(2)}-2p_{n+1,n-k+1}^{(2)}+p_{n,n-k}^{(2)}\geqq 0 \quad (k=2,\ldots,n),$

(a_4) $p_{n+2,1}^{(2)}-2p_{n+1,0}^{(2)}\geqq 0,$

(a_5) $p_{n+2,0}^{(2)}\geqq 0.$

hold.

In present paper the necessary and sufficient condition for matrix T preserving (p,p)-convexity, $p>0$, i.e. $(1,q)$-convexity $q>0$, $q\neq 1$, will be given.

The difference equation

$$\Delta_{p,q}a_n=0,\ p>0,\ q>0, \tag{1.7}$$

has the characteristic equation

$$\lambda^2-(p+q)\lambda+pq=0. \tag{1.8}$$

If p and q are real, two essentially different cases will be considered:

1° $p=q$. In this case, the basic system of solutions of the equation (1.7) is $(p^n, np^n)\,(n=1,2,\ldots)$;

2° $p\neq q$, then the basic system is $(p^n, q^n)\,(n=1,2,\ldots)$.

2. In this part of paper we will consider $p=q$, $p>0$, i.e. the difference operator

$$\Delta_{p,q}a_n=\Delta_p^2 a_n=a_{n+2}-2p\,a_{n+1}+p^2 a_n \quad (n\in \mathrm{N}_0). \tag{2.1}$$

The formal equality $\Delta_{p,p}a_n=\Delta_p^2 a_n$ is a consequence of the iteration of the first order difference operator

$$\Delta_p a_n=a_{n+1}-pa_n. \tag{2.2}$$

Indeed, we have $\Delta_{p,p}=\Delta_p(\Delta_p)=\Delta_p^2$.

Now, we will prove the following lemmas.

Lemma 1. *For every* $m=1, 2, 3, \dots$ *the sequence* $(u_i^m)\,(i=1, 2, \dots)$ *defined by*

$$u_i^m=\begin{cases} p^i, & 1\leqq i\leqq m\\ (i-m)p^i, & i\geqq m+1,\ p>0,\end{cases}$$

is (p, p)*-convex, i. e.* $\Delta_p^2 u_i^m \geqq 0$ $(i=1, 2, \dots;\ m=1, 2, 3, \dots)$.

Proof. It immediately follows for sequence (2.3) that for every $m=1, 2, 3, \dots$ we have

$$\Delta_p^2 u_i^m=\begin{cases} p^{m+2}, & i=m\\ 0, & i\neq m,\end{cases}$$

and, if we take into account the condition $p>0$, we have the proof of the lemma.

Now, for triangular matrix $(p_{nk})\,(k=0, 1, \dots, n;\ n=0, 1, 2, \dots)$, we put

$$P_{n,k}=\sum_{i=1}^{k} p^{n-i}p_{ni},\quad Q_{n,k}=\sum_{i=0}^{k} P_{n,i}. \tag{2.5}$$

Also, we write

$b_0=a_0$, $b_1=\Delta_p a_0=a_1-pa_0$, $b_k=\Delta_p^2 a_{k-2}=a_k-2pa_{k-1}+p^2 a_{k-2}\,(k=2, 3, \dots, n)$.

We have the following lemma:

Lemma 2. *For every* $n=0, 1, 2, \dots$ *and* $s_n(a)$ *defined by* (1.5), *the following identity holds*

$$s_n(a)=P_{n,n}b_0+\frac{1}{p}Q_{n,n-1}b_1+\frac{1}{p^2}Q_{n,n-2}b_2+\cdots+\frac{1}{p^n}Q_{n,0}b_n, \tag{2.6}$$

where a is an arbitrary real sequence.

Proof. Determine the coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$ in

$$a_n=\alpha_0 b_0+\alpha_1 b_1+\cdots+\alpha_n b_n \tag{2.7}$$

for an arbitrary $a_0, a_1, \dots, a_n$. Putting in (2.7) $a_i=p^{i+1}\,(i=0, 1, 2, \dots)$ we have $b_0=a_0=p$, $b_k=0\,(k=1, 2, \dots, n)$, because the sequence p^{i+1} is (p, p)-convex. Put $a_i=u_i^k\,(i=0, 1, \dots, n)$. In virtue of lemma 1, for fixed k, we have $b_k=p^k$ and $b_i=0$ $(i\neq k)$ and we get $\alpha_k=(n-k+1)p^{n-k}\,(k=1, 2, \dots, n)$.

In such a manner, (2.7) becomes

$$a_n=p^n b_0+np^{n-1}b_1+(n-1)p^{n-2}b_2+\cdots+2pb_{n-1}+b_n,$$

so we have

$$\begin{aligned} s_n=&(p_{nn}+pp_{nn-1}+p^2 p_{nn-2}+\cdots+p^n p_{n0})\,b_0\\ &+(p_{nn-1}+2pp_{nn-2}+\cdots+np^{n-1}p_{n0})\,b_1\\ &+(p_{nn-2}+2pp_{nn-3}+\cdots+(n-1)p^{n-2}p_{n0})\,b_2\\ &+\cdots+(2pp_{n0}+p_{n1})\,b_{n-1}+p_{n0}b_n,\end{aligned}$$

wherefrom (2.6) follows.

Now, we have the following theorem:

Theorem 1. *The necessary and cufficient condition that the implication*

$$(a_{n+2}-2pa_{n+1}+p^2 a_n\geqq 0)\ \Rightarrow\ (s_{n+2}-2ps_{n+1}+p^2 s_n\geqq 0) \tag{2.8}$$

($n \in \mathbf{N}_0$, $p>0$) *is valid, for every sequence* (a_n), *where the sequence* (s_n) *is given by* (1.5), *is that the following conditions hold:*

(b_1) $P_{n+2,\,n+2}-2\,p\,P_{n+1,\,n+1}+p^2 P_{n,\,n}=0$,

(b_2) $Q_{n+2,\,n+1}-2\,p\,Q_{n+1,\,n-1}+p^2 Q_{n,\,n-1}=0$,

(b_3) $Q_{n+2,\,n-k+2}-2\,p\,Q_{n+1,\,n-k+1}+p^2 Q_{n,\,n-k}\geqq 0$ $(k=2, 3, \ldots, n)$,

(b_4) $Q_{n+2,\,1}-2\,p\,Q_{n+1,\,0}\geqq 0$,

(b_5) $Q_{n+2,\,0}\geqq 0$ (i.e. $p_{n0}\geqq 0$).

Proof. Prior to proceeding to the proof of this theorem we will establish that on the base of (2.6) we have

$$\Delta_p^2 S_n=(P_{n+2,\,n+2}-2\,p\,P_{n+1,\,n+1}+p^2 P_{n,\,n})\,b_0 \tag{2.9}$$
$$+\frac{1}{p}(Q_{n+2,\,n+1}-2\,p\,Q_{n+1,\,n}+p^2 Q_{n,\,n-1})\,b_1$$
$$+\sum_{k=2}^{n}\frac{1}{p^k}(Q_{n+2,\,n-k+2}-2\,p\,Q_{n+1,\,n-k+1}+p^2 Q_{n,\,n-k})\,b_k$$
$$+\frac{1}{p^{n+1}}(Q_{n+2,\,1}-2\,p\,Q_{n+1,\,0})\,b_{n+1}+\frac{1}{p^{n+2}}Q_{n+2,\,0}\,b_{n+2}.$$

The conditions (b_1) — (b_5) are necessary.

The identity (2.6) holds for an arbitrary sequence (a_n) $(n\in\mathbf{N}_0)$. If we take $a_n=p^{n+1}$ $(n=0, 1, 2, \ldots)$, we have $\Delta_p^2 a_n\geqq 0$. In virtue of the implication (2.8), taking into account that $b_k=0$ $(k=1, 2, \ldots, n)$, we get

$$\Delta_p^2 s_n=P_{n+2,\,n+2}-2\,p\,P_{n+1,\,n+1}+p^2 P_{n,\,n}\geqq 0. \tag{2.10}$$

Otherwise, the sequence $a_n=-p^{n+1}$ $(n\in\mathbf{N}_0)$ is also (p, p)-convex and for this reason

$$\Delta_p^2 s_n=-(P_{n+2,\,n+2}-2\,p\,P_{n+1,\,n+1}+p^2 P_{n,\,n})\geqq 0. \tag{2.11}$$

The inequalities (2.10) and (2.11), taken together, give the condition (b_1).

Now, let $a_n=(n+1)\,p^{n+1}$ $(n\in\mathbf{N}_0)$. We have $b_1=p^2$, $b_k=0$ $(k=2, \ldots, n)$ and $\Delta_p^2 a_n\geqq 0$, so that

$$\Delta_p^2 s_n=\frac{1}{p}(Q_{n+2,\,n+1}-2\,p\,Q_{n+1,\,n}+p^2 Q_{n,\,n-1})\,p^2\geqq 0. \tag{2.12}$$

On the other hand, for $a_n=-(n+1)\,p^{n+1}$ $(n\in\mathbf{N}_0)$, we have $b_1=-p^2$, $b_k=$ $=0$ $(k=2, \ldots, n)$ and $\Delta_p^2 a_n\geqq 0$. Thus, we get

$$\Delta_p^2 s_n=\frac{1}{p}(Q_{n+2,\,n+1}-2\,p\,Q_{n+1,\,n}+p^2 Q_{n,\,n-1})\,(-p^2)\geqq 0,$$

which together with (2.12) and for $p>0$ gives (b_2).

With the conditions (b_1) and (b_2), (2.9) becomes

$$(2.13) \qquad \Delta_p^2 s_n = \sum_{k=2}^{n} \frac{1}{p^2} (Q_{n+2,\,n-k+2} - 2\,p\,Q_{n+1,\,n-k+1} + p^2 Q_{n,\,n-k})\,b_k + \frac{1}{p^{n+1}} (Q_{n+1,\,1} - 2\,p\,Q_{n+1,\,0})\,b_{n+1} + \frac{1}{p^{n+2}} Q_{n+2,\,0}\,b_{n+2}.$$

Further, choose $a = u^m$, where $u^m = (u_i^m)$ is given by (2.3). Let,

$$(2.14) \qquad b_k^m = \Delta_p^2 u_{k-1}^m \quad (k = 2, 3, \dots,\ m = 1, 2, \dots),$$

Thus, for fixed $m = k-1$ we have $b_k^{k-1} = p^{k+1} \geqq 0$, so it will be

$$\Delta_p^2 s_n = \frac{1}{p^{n+1}} (Q_{n+2,\,n-k+2} - 2\,p\,Q_{n+1,\,n-k+1} + p^2 Q_{n,\,n-k})\,p^{k+1} \geqq 0$$

wherefrom, for $k = 2, 3, \dots, n$ we have the condition (b_3).

Now, (2.13) becomes

$$\Delta_p^2 s_n = \frac{1}{p^{n+1}} (Q_{n+1,\,2} - 2\,p\,Q_{n+1,\,0})\,b_{n+1} + \frac{1}{p^{n+2}} Q_{n+2,\,0}\,b_{n+2}.$$

Further, if we put $k = n+1$, $m = n$ in (1.14), we have $b_{n+1}^n = \Delta_p^2 u_n^n = p^{n+1}$, $b_{n+2}^n = 0$. On the basis of (p, p)-convexity of sequence u_k^n $(k = 1, 2, \dots)$ we have that the sequence s_n is also (p, p)-convex and it is equivalent to the condition (b_4).

If we put $a_n = u_n^{n+1}$, the last condition (b_5) is obtained. Hence $b_{n+2}^{n+1} = p^{n+3} \geqq 0$.

The conditions (b_1) — (b_5) are sufficient.

If (b_1) — (b_5) hold, it is clear from (2.9) that $\Delta_p^2 a \geqq 0 \Rightarrow \Delta_p^2 s_n \geqq 0$. $b_0 = a_0$ and $b_1 = a_1 - pa_0$ may have an arbitrary sign. This completes the proof.

REMARK. For $p = 1$ we obtain OSEKI's theorem A.

3. In this section, we consider the case when the roots of the characteristic equation (1.8) are 1 and $q\,(q > 0,\ q \neq 1)$. Corresponding difference operator will be

$$(3.1) \qquad \Delta_{1,\,q}\, a_n = a_{n+2} - (1+q)\,a_{n+1} + q a_n \quad (n \in \mathbf{N}_0).$$

We will prove the following statements:

Lemma 3. *For every $m = 1, 2, 3, \dots$ the sequence*

$$(3.2) \qquad v_i^m = \begin{cases} 1, & 0 \leqq i \leqq m-1 \\ q^{i-m+1}, & i \geqq m, \end{cases}$$

is $(1, q)$-convex for $q > 1$.

Proof. We have

$$(3.3) \qquad \Delta_{1,\,q}\, v_i^m = \begin{cases} q-1, & i = m-2 \\ 0, & i \neq m-2,\ m = 2, 3, 4, \dots, \end{cases}$$

and it is obvious that $\Delta_{1,\,q} v_i^m \geqq 0$ for $q > 1$.

Now, for matrix (p_{nk}) $(k=0, 1, \dots, n;\ n=0, 1, 2, \dots)$ let

(3.4) $$R_{n,k}=\sum_{i=1}^{k} p_{n,i}, \qquad S_{n,k}=\sum_{i=0}^{k} q^{k-i} R_{n,i},$$

and, for the sequence (a_n) let

(3.5) $$c_0=a_0,\ c_1=a_1-a_0,\ c_k=\Delta_{1,q}\,a_{k-2}\ (k=2, \dots, n).$$

Then we have

Lemma 4. *For every* $n=0, 1, 2, \dots$ *the identity*

(3.6) $$s_n=R_{n,n}c_0+S_{n,n-1}c_1+S_{n,n-2}c_2+\cdots+S_{n,0}c_n.$$

holds.

Proof. Suppose that

(3.7) $$a_n=\beta_0 c_0+\beta_1 c_1+\beta_2 c_2+\cdots+\beta_n c_n \quad (n\in \mathbf{N}_0).$$

Putting in (3.7) $a_n=1\ (n\in\mathbf{N}_0)$ we obtain $\beta_0=1$. If, for $k=1, 2, \dots, n$, we chose $a_i=v_i^k\ (i=0, 1, \dots, n)$, according to lemma 3, it follows that

$$c_k=\Delta_{1,q}v_{k-2}^{k}=q-1,\ c_i=0\ (i\neq k).$$

Hence $q^{n-k+1}=1+\beta_k(q-1)$ wherefrom $\beta_k=\dfrac{q^{n-k+1}}{q-1}$ $(k=1, 2, \dots, n)$ follows.

Thus, we have $a_n=c_0+\dfrac{q^n-1}{q-1}c_1+\dfrac{q^{n-1}-1}{q-1}c_2+\cdots+\dfrac{q-1}{q-1}c_n$, and

$$s_n=(p_{n0}+p_{n1}+\cdots+p_{nn})c_0+\left(p_{nn-1}+\frac{q^2-1}{q-1}p_{nn-2}+\cdots+\frac{q^n-1}{q-1}p_{n0}\right)c_1$$

$$+\left(p_{nn-2}+\frac{q^2-1}{q-1}p_{nn-3}+\cdots+\frac{q^{n-1}-1}{q-1}p_{n0}\right)c_2+\cdots+\left(p_{nn}+\frac{q^2-1}{q-1}p_{n0}\right)c_{n-1}+p_{n0}c_n.$$

Using the identity $\dfrac{q^{m+1}-1}{q-1}=q^m+q^{m-1}+\cdots+q+1$ $(q\neq 1)$, according the notations (3.4), we have

$$s_n=R_{n,n}c_0+(R_{n,n-1}+q R_{n,n-2}+q^2 R_{n,n-3}+\cdots+q^{n-1}R_{n,0})c_1$$
$$+(R_{n,n-2}+q R_{n,n-3}+\cdots+q^{n-2}R_{n,0})c_2$$
$$+\cdots+(R_{n,1}+q R_{n,0})c_{n-1}+R_{n,0}c_n,$$

wrerefrom the statement of lemma follows.

Now, we will prove the following theorem:

Theorem 2. *The necessary and sufficient condition that the implication*

(3.8) $$(a_{n+2}-(1+q)a_{n+1}qa_n\geqq 0)\ \Rightarrow$$
$$\Rightarrow\ (s_{n+2}-(1+q)s_{n+1}+qs_n\geqq 0)\ (n\in\mathbf{N}_0,\ q>0,\ q\neq 1)$$

is valid for every sequence (a_n), *where* (s_n) *is given by* (1.5), *is that the conditions*

(c_1) $R_{n+2,n+2}-(1+q)R_{n+1,n+1}+q R_{n,n}=0,$

(c_2) $S_{n+2,n+1}-(1+q)S_{n+1,n}+qS_{n,n-1}=0$,

(c_3) $S_{n+2,n-k+2}-(1+q)S_{n+1,n-k+1}+qS_{n,n-k}\geqq 0 \quad (k=2, \dots, n-1)$,

(c_4) $S_{n+2,1}-(1+q)S_{n+1,0}\geqq 0$,

(c_5) $S_{n+2,0}\geqq 0 \quad$ (i.e. $p_{n0}\geqq 0$).

hold.

Proof. From (3.6) we have

$$\begin{aligned}\Delta_{1,q}s_n=&(R_{n+2,n+2}-(1+q)R_{n+1,n+1}+qR_{n,n})c_0\\ &+(S_{n+2,n+1}-(1+q)S_{n,n+1}+qS_{n,n-1})c_1\\ &+\sum_{k=2}^{n}(S_{n+2,n-k+2}-(1+q)S_{n+1,n-k+1}+qS_{n,n-1})c_k\\ &+(S_{n+2,1}-(1+q)S_{n+1,0})c_{n+1}+S_{n+2,0}c_{n+2}.\end{aligned} \tag{3.9}$$

The conditions (c_1) — (c_5) are necessary.

On the basis of the assumption of theorem, the implication (3.8) holds. The sequences $a_n=+1\,(n\in\mathbf{N}_0)$ and $a_n=-1\,(n\in\mathbf{N}_0)$ satisfy the condition $\Delta_{1,q}a_n\geqq 0$, so that $\Delta_{1,q}s_n\geqq 0$, i.e. $\pm(R_{n+2,n+2}-(1+q)R_{n+1,n+1}+qR_{n,n})\geqq 0$, which is equivalent to the condition (c_1).

Now, (3.9) reduces to

$$\begin{aligned}\Delta_{1,q}s_n=&(S_{n+1,n-1}-(1+q)S_{n+1,n}+qS_{n,n-1})c_1\\ &+\sum_{k=2}^{n}(S_{n+2,n-k+2}-(1+q)S_{n+1,n-k+1}+qS_{n,n-k})c_k\\ &+(S_{n+2,1}-(1+q)S_{n+1,0})c_{n+1}+qS_{n+2,0}c_{n+2}.\end{aligned} \tag{3.10}$$

Sequences $a_n'=q^n\,(n\in\mathbf{N}_0)$ and $a_n''=-q^n\,(n\in\mathbf{N}_0)$ satisfied the conditions $\Delta_{1,q}a_n'=\Delta_{1,q}a_n''=0$, i.e. they are $(1, q)$-convex. Then it must be $\Delta_{1,q}s_n\geqq 0$. Besides that, for sequences a_n' and a_n'' will be $c_i'=c_i''=0\,(i=2, 3, \dots)$, $c_1'=q-1$ and $c_1''=1-q$, so that

$$\Delta_{1,q}s_n=(S_{n+2,n-1}-(1+q)S_{n+1,n}+qS_{n,n-1})(q-1)\geqq 0,$$

and

$$\Delta_{1,q}s_n=(S_{n+2,n-1}-(1+q)S_{n+1,n}+qS_{n,n-1})(1-q)\geqq 0,$$

wherefrom $(S_{n+2,n-1}-(1+q)S_{n+1,n}+qS_{n,n-1})(q-1)=0$. Therefore, for $q\neq 1$, the condition (c_2) is valid.

Now, (3.10) reads

$$\begin{aligned}\Delta_{1,q}s_n=&\sum_{k=2}^{n}(S_{n+2,n-k+2}-(1+q)S_{n+1,n-k+1}+qS_{n,n-k})c_k\\ &+(S_{n+2,1}-(1+q)S_{n+1,0})c_{n+1}+S_{n+2,0}c_{n+2}.\end{aligned} \tag{3.11}$$

We shall consider two cases

1° $q>1$. Put $a_n=v_n^k (k=2, 3, \ldots, n+2)$. For this sequence we have $\Delta_{1,q} v_{k-2}^k=q-1>0$, $\Delta_{1,q} v_j^k=0 \, (j\neq k-2)$ (see lemma 3), i.e. it is (1, q)-convex. Therefore

$$\Delta_{1,q} s_n=(S_{n+2,n-k+2}-(1+q)\, S_{n+1,n-k+1}+q\, S_{n,n-k})\,(q-1)\geqq 0,$$

i.e. the conditions (c_3) — (c_5) are satisfied.

2° $0<q<1$. If we select $a_n=-v_n^k$, $\Delta_{1,q}(-v_{k-2}^k)=(1-q)>0$, $\Delta_{1,q} v_j^k= =0 \, (j\neq k-1)$ i.e. (a_n) is (1, q)-convex, then we rederive (c_3) — (c_5).

Sufficiency of the conditions follows from (3.9). It is evident that a_0 and a_1-a_0 may have an arbitrary sign.

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Elektronski fakultet
18 000 Niš, Jugoslavija

O GENERALNOJ KONVEKSNOSTI KOJA ČUVA MATRIČNU TRANSFORMACIJU

Ljubiša M. Kocić

U radu su dati potrebni i dovoljni uslovi za realnu trougaonu matricu (p_{nk}) $(0\leqq k\leqq n;$ $n=0, 1, \ldots)$ tako da važi implikacija

$$(a_{n+2}+(p+q)\, a_{n+1}+pqa_n\geqq 0) \Rightarrow (s_{n+2}+(p+q)\, s_{n+1}+pqs_n\geqq 0 \quad (n=0, 1, \ldots),$$

gde je $s_n=\sum_{k=0}^{n} p_{nn-k}\, a_k$. Pritom su razmatrani slučajevi 1° $p=q>0$ i 2° $p=1$, $q>0$, $q\neq 1$.