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# 725. INEQUALITIES FOR POLYGONS*

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In this paper we generalize a number of known triangle inequalities by using more general functions of the sides of the triangle and by replacing the triangle by a polygon.

Our first result is:

**Theorem 1.** *If* $(n-1)s$ *is the perimeter of an n-gon with sides* $x_j$ *such that* $x_j \leqq s$ $(1 \leqq j \leqq n)$ *and* $F(x)$ *is strictly convex for* $x \geqq 0$, *then*

$$\sum_{i=1}^{n} F(x_i) \leqq \sum_{i=1}^{n} F((n-1)(s-x_i)) \tag{1}$$

*with equality if and only if* $x_1 = x_2 = \cdots = x_n$. *The inequality is reversed if F is concave.*

***Proof.*** Assume without loss of generality that $x_1 \geqq x_2 \geqq \cdots \geqq x_n$. Then the vector $\{(n-1)(s-x_n), (n-1)(s-x_{n-1}), \ldots, (n-1)(s-x_1)\}$ majorizes the vector $\{x_1, x_2, \ldots, x_n\}$. Inequality (1) now follows immediately from the Majorization inequality [**1**, p. 112].

By letting $F(x) = \log x$ for $x > 0$ in (1), we obtain

**Corollary 1.** $x_1 x_2 \cdots x_n \geqq (n-1)^n (s-x_1)(s-x_2) \cdots (s-x_n)$.

The later result had been obtained previously by D. D. ADAMOVIĆ in answer to a problem of D. S. MITRINOVIĆ [**1**, p. 209]. Also, by letting $F(x) = x^{-t}$ and $n = 3$ in (1), we obtain inequalities 6.19 from [**2**], i. e.,

$$h_a{}^t + h_b{}^t + h_c{}^t \geqq r_a{}^t + r_b{}^t + r_c{}^t, \qquad -1 < t < 0, \tag{2}$$

and the reversed inequality if $t > 0$ or $t < -1$. Note here that $ah_a = 2F$, $(s-a)r_a = F$, etc., where $F$ is the area of triangle $(a, b, c)$.

Under the same conditions as Theorem 1, we obtain in a similar way

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**Theorem 2.**

$$\sum_{i=1}^{n} F(x_i) \geqq \sum_{i=1}^{n} F\left(s-\frac{x_i}{n-1}\right) \tag{3}$$

*with equality if and only if* $x_1=x_2=\cdots=x_n$.

By letting $F(x)=-\log(n-1)x$, we obtain

**Corollary 2.** $(n-1)^n x_1 x_2 \cdots x_n \leqq (p-x_i)(p-x_2)\cdots(p-x_n)$ *where* $p=(n-1)s=x_1+x_2+\cdots+x_n$.

For $n=3$, Corollary 2 reduces to inequality 1.4 from **[2]**. This corollary holds for arbitrary non-negative numbers $x_i$ and can also be obtained by applying the A. M.—G. M. inequality to each of the terms $p-x_i$, i. e.,

$$\frac{p-x_i}{n-1} \geqq \left(\frac{x_1 x_2 \cdots x_n}{x_i}\right)^{1/(n-1)}.$$

Again under the same conditions as Theorem 1, we obtain in a similar way the following triangle and quadrilateral inequalities

$$\sum_{i=1}^{3} F(p-x_i) \geqq \sum_{i=1}^{3} F\left(\frac{p+x_i}{2}\right), \tag{4}$$

$$\sum_{i=1}^{4} F(p-x_i) \geqq \sum_{i=1}^{4} F\left(\frac{p+x_i+x_{i+1}}{2}\right), \quad (x_5=x_1). \tag{5}$$

Since $p-x_i \geqq 0$, the $x_i$ can be arbitrary non-negative numbers and so need not form a polygon.

For the next result, we have the same conditions as in Theorem 1 plus $F'(x)\geqq 0$, $F''(x)\geqq 0$ and $\lambda \geqq 1$.

**Theorem 3.**

$$\sum_{i=1}^{n} F\left(\frac{x_i}{\lambda s-x_i}\right) \geqq n F\left(\frac{n-1}{n(\lambda-1)+1}\right). \tag{6}$$

***Proof.*** Let $y(x)=F(t)$ where $t=x/(\lambda s-x)$. Since

$$y''(x)=\lambda^2 s^2(\lambda s-x)^{-4} F''(t)+2\lambda s(\lambda s-x)^{-3} F'(t), \tag{7}$$

we have $y''(x)\geqq 0$ and from JENSEN's inequality we have (6) with equality if and only if $x_1=x_2=\cdots=x_n$.

Letting $n=3$, $\lambda=1$, we obtain the following two special cases of Theorem 2:

$$F\left(\frac{2r}{h_a-2r}\right)+F\left(\frac{2r}{h_b-2r}\right)+F\left(\frac{2r}{h_c-2r}\right)\geqq 3F(2), \tag{8}$$

$$F\left(\frac{2r_a}{h_a}\right)+F\left(\frac{2r_b}{h_b}\right)+F\left(\frac{2r_c}{h_c}\right)\geqq 3F(2). \tag{9}$$

Note that here $\frac{2r}{h_a-2r}=\frac{a}{s-a}=\frac{2r_a}{h_a}$, etc., and (8) and (9) are the same. Letting $F(x)=x$ in (8) and (9) gives inequalities 6.21 and 6.28 in [2].

Our next result is a complementary inequality to Theorem 3.

**Theorem 4.** *If $F(0)=0$, $F(x)\geqq 0$, $F''(x)\geqq 0$, $ks$=perimeter $p$ of an $n$-gon of sides $x_i$ with $x_i\leqq s\,(1\leqq i\leqq n)$ and $\lambda>1$, then*

$$\sum_{i=1}^{n} F\left(\frac{x_i}{\lambda s-x_i}\right)\leqq kF\left(\frac{1}{\lambda-1}\right). \tag{10}$$

A proof follows directly from the known inequality [3]

$$\frac{1}{n}\sum_{i=1}^{n} G(x_i)\leqq\frac{M-\bar{x}}{M-m}G(m)+\frac{\bar{x}-m}{M-m}G(M) \tag{11}$$

where $G(x)$ is a convex function on $[a, b]$, $[m, M]\subseteq[a, b]$, $m\leqq x_i\leqq M\,(1\leqq i\leqq n)$ and $\bar{x}=(x_1+x_2+\cdots+x_n)/n$. Here, $m=0$, $M=s$, $\bar{x}=ks/n$ giving

$$\sum_{i=1}^{n} G(x_i)\leqq(n-k)\,G(0)+kG(s). \tag{12}$$

For $k=1$, (12) reduces to an inequality of Petrović [1, p. 22] while for $k=n-1$ we get a result from Pavlović [4]. Letting $G(x)=F(x/(\lambda s-x))$ in (12) we obtain (10). If we now let $F(x)=x$, $k=n-1=\lambda$ in (7) and (10), we obtain inequalities 16.5 from [2] (also see [4]).

## REFERENCES

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## NEJEDNAKOSTI ZA POLIGON

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U radu su date generalizacije niza poznatih nejednakosti za trougao. Pri tome su korišćene opštije funkcije strana trougla, a trougao je zamenjen poligonom.