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724. SOME REMARKS ON THE GENERAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS*

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1. Except in the case of linear equations, the *general solution* of a differential equation is a very delicate concept. The general solution, taken literally, means the solution which contains all the solutions of the considered equation, and this applies to all equations, not only to differential equations. However, such a definition is rarely ever given in books on differential equations. Careful writers avoid to use the term „general solution", except for linear equations, but they rather use expressions such as, „solution containing one arbitrary constant", etc. Nevertheless, one can often find that the general solution of an n-th order equation is identified with the solution containing n arbitrary constants. Since some simple equations show that this definition is not correct, the concept „singular solution" is introduced, meaning the solution which is not contained in the „general solution".

EXAMPLE 1.1. For CLAIRAUT's equation $y=xy'+(y')^2$ it is said that it possesses the general solution $y=Cx+C^2$ (C arbitrary constant) and the singular solution $y=-x^2/4$.

In his book [1] GOURSAT writes: In order to recognize whether a solution is general, it is not enough to count the arbitrary elements which appear in the solution.

EXAMPLE 1.2. The solution

$$y=C_1+C_2x+C_3x^2 \quad (C_1,\ C_2,\ C_3 \text{ arbitrary constants}) \tag{1.1}$$

of the third order equation

$$yy''y'''-(y')^2y'''=0 \tag{1.2}$$

contains three arbitrary constants, but it is not its general solution. Indeed, the eq. (1 2) is also satisfied by any function of the form $y=Ae^{Bx}$ (A, B arbitrary constants), and such functions are not contained in (1.1).

In further text the term *general solution* will be used to denote the solution containing all the solutions.

2. Consider the diferential equation

$$F(x,y,y',\dots,y^{(n)})=0. \tag{2.1}$$

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To find the general solution of (2.1) is equivalent to finding the set $S \subset D_n$ (D_n is the set of all n times differentiable functions) such that

$$F(x, y, y', \dots, y^{(n)}) = 0 \Leftrightarrow y \in S.$$

Since, by definition, the set S contains all the solutions of the eq. (2.1) we could call S the general solution of (2.1). However, in order to conform with the usual practice, we shall say that the general solution of (2.1) is defined by a surjection P mapping D_n onto S, i.e. we say that general solution of (2.1) is given by

$$y(x) = Pu(x) \qquad (u \in D_n \text{ is arbitrary}). \tag{2.2}$$

Naturally, we should not expect to be able to express P by one „analytic expression".

Formula (2.2) for the general solution of (2.1) implies that the general solution is defined by a fixed mapping P and an arbitrary function $u \in D_n$. This is in accordance with the general approach suggested by PREŠIĆ ([2], [3]), but, as we shall see, in case of differential equations, the mapping P will be such that $Pu(x)$ can be made to depend on arbitrary constants, instead of one arbitrary function.

EXAMPLE 2.1. For the equation

$$yy'' - (y')^2 = 0 \tag{2.3}$$

the following equivalence holds:

$$yy'' - (y')^2 = 0 \Leftrightarrow y \in \{x \mapsto Ae^{Bx} \mid A, B \in \mathbf{R}\} = S.$$

A surjection $P: D_2 \to S$ is defined by

$$Pu(x) = u(a)\, e^{u'(a)x} \qquad (a \text{ is a fixed point})$$

and hence the general solution of (2.3) is given by

$$y(x) = u(a)\, e^{u'(a)x} \qquad (u \in D_2 \text{ is arbitrary}),$$

or, equivalently $y(x) = Ae^{Bx}$, where $A = u(a)$, $B = u'(a)$ are arbitrary constants.

EXAMPLE 2.2. For CLAIRAUT's equation $y = xy' + (y')^2$ we have

$$y = xy' + (y')^2 \Leftrightarrow y \in S = \{x \mapsto Cx + C^2 \mid C \in \mathbf{R}\} \cup \{x \mapsto -x^2/4\}.$$

A surjection P is defined by

$$Pu(x) = \begin{cases} u(a)\, x + u(a)^2, & u(x) \not\equiv 0 \\ -x^2/4 & , \ u(x) \equiv 0 \end{cases} \tag{2.4}$$

giving the known solutions $y(x) = Cx + C^2$ and $y = -x^2/4$, where $C = u(a)$ is an arbitrary constant. The general solution, however, is not $y = Cx + C^2$, but $y(x) = Pu(x)$, where P is given by (2.4).

EXAMPLE 2.3. For the equation $(y - xy')(y' - 1) = 0$ we have

$$(y - xy')(y' - 1) = 0 \Leftrightarrow y \in S = \{x \mapsto Cx \mid C \in \mathbf{R}\} \cup \{x \mapsto x + C \mid C \in \mathbf{R}\}.$$

A surjection $P: D_1 \to S$ is defined by

$$Pu(x) = \begin{cases} u(a)\, x, & u(x) = px + q \\ x + u(a), & u(x) \neq px + q \end{cases} \qquad (p, q \in \mathbf{R}),$$

and the general solution of the considered equation is given by $y(x)=Pu(x)$, where $u(a)$ is again denoted by C (arbitrary constant).

EXAMPLE 2.4. For the equation

$$y^{(n)}+a_1 y^{(n-1)}+\cdots+a_n y=0 \qquad (a_k \text{ given constants}) \tag{2.5}$$

CAUCHY [4] (see also [5], pp. 202—204) gave the following solution:

$$y(x)=\sum \operatorname{Res} \frac{u(z)}{g(z)} e^{zx},$$

where u is an arbitrary analytic function whose zeroes do not coincide with the zeroes of the characteristic polynomial $g(z)=z^n+a_1 z^{n-1}+\cdots+a_n$. The summation is taken over all the zeroes of g. In this case a surjection P is defined by

$$Pu(x)=\sum \operatorname{Res} \frac{u(z)}{g(z)} e^{zx}.$$

Suppose, for instance, that $\alpha_1, \ldots, \alpha_n$ are simple zeroes of g. Then

$$\operatorname*{Res}_{z=\alpha_k} \frac{u(z)}{g(z)} e^{zx}=\lim_{z\to\alpha_k}(z-\alpha_k)\frac{u(z)}{g(z)} e^{zx}=\frac{u(\alpha_k)}{g'(\alpha_k)} e^{\alpha_k x},$$

and hence,

$$Pu(x)=\sum_{k=1}^{n} \frac{u(\alpha_k)}{g'(\alpha_k)} e^{\alpha_k x}.$$

Putting $\frac{u(\alpha_k)}{g'(\alpha_k)}=C_k$ $(k=1, \ldots, n)$, where $C_1, \ldots, C_n$ are arbitrary constants, we arrive at the standard form of the solution of (2.5).

3. An elegant way to prove that a solution of an equation (E) is general is to replace the eq. (E) by an equivalent *reproductive* equation.

Let $f: S \to S$ where S is a nonempty set. We say that the equation

$$x=f(x) \tag{3.1}$$

is reproductive if $f^2=f$, i.e. $f(f(x))=f(x)$ for all $x \in S$. It is easily verified that the general solution of the reproductive equation (3.1) is $x=f(t)$, where $t \in S$ is arbitrary.

The above definition and the result are due to PREŠIĆ [2]. He also proved that for any equation $x=g(x)$, where $g: S \to S$, there exists an equivalent reproductive equation, i.e.

$$x=g(x) \Leftrightarrow x=f(x), \text{ where } f^2=f.$$

It that case $x=f(t)$ is the general solution of the equation $x=g(x)$, where $t \in S$ is arbitrary.

A convenient straightforward modification of the above is as follows: If

$$x=g(x) \Leftrightarrow x=f_1(x) \vee \cdots \vee x=f_n(x),$$

where $f_k^2=f_k$ $(k=1, \ldots, n)$, then all the solutions of the equation $x=g(x)$ are given by

$$x=f_1(t) \vee \cdots \vee x=f_n(t), \text{ where } t \in S \text{ is arbitrary}.$$

We shall show here how the equations from Examples 2.1, 2.2 and 2.3 can be put into reproductive form.

The following auxilliary result is necessary: The equation $y'=0$ is equivalent to the reproductivr equation $y(x)=y(a)$, where a is fixed. This follows directly from the meanvalue theorem.

EXAMPLE 3.1. For $y \neq 0$ we have

$$yy''-(y')^2=0 \Leftrightarrow \left(\frac{y'}{y}\right)'=0 \Leftrightarrow \frac{y'(x)}{y(x)}=\frac{y'(a)}{y(a)} \Leftrightarrow y'(x)-\frac{y'(a)}{y(a)}\,y(x)=0 \Leftrightarrow$$

$$\left(y(x)\exp\left(-\frac{y'(a)}{y(a)}\,x\right)\right)'=0 \Leftrightarrow y(x)\exp\left(-\frac{y'(a)}{y(a)}\,x\right)=y(a)\exp\left(-\frac{y'(a)}{y(a)}\,a\right) \Leftrightarrow$$

$$y(x)=y(a)\exp\left(\frac{y'(a)}{y(a)}(x-a)\right).$$

The last equation is reproductive, and hence its general solution is given by

$$y(x)=F(a)\exp\left(\frac{F'(a)}{F(a)}(x-a)\right),$$

where $F \in D_2$ is an arbitrary function.

Putting $F(a)\exp\left(-\frac{F'(a)}{F(a)}\,a\right)=A$, $\frac{F'(a)}{F(a)}=B$, we obtained the standard form of the general solution: $y=Ae^{Bx}$, where A and B are arbitrary constants.

EXAMPLE 3.2. From $y=xy'+(y')^2$ follows $y(0)=y'(0)^2$ and, differentiating, $y''(x+2y')=0$.

If $y''=0$ then $y'''=y^{(4)}=\cdots=0$.

If $x+2y'=0$, then $y'(0)=0$ and $1+2y''=0$, $y'''=y^{(4)}=\cdots=0$. Hence, developping y into TAYLOR's series, we find

$$y=xy'+(y')^2 \Leftrightarrow y(x)=y'(0)^2+y'(0)\,x \vee y(x)=-\frac{1}{4}x^2. \tag{3.2}$$

Since both equations on the right hand side of (3.2) are reproductive, all the solutions of the considered CLAIRAUT's equation are given by

$$y(x)=F'(0)^2+F'(0)\,x \vee y(x)=-\frac{1}{4}x^2,$$

where $F \in D_1$ is an arbitrary function, or

$$y=C^2+Cx \vee y=-\frac{1}{4}x^2,$$

where $C=F'(0)$ is an arbitrary constant.

EXAMPLE 3.3. We have $(y-xy')(y'-1)=0 \Leftrightarrow \left(\frac{y}{x}\right)'=0 \vee (y-x)'=0$

$$\Leftrightarrow \frac{y(x)}{x}=\frac{y(a)}{a} \vee y(x)-x=y(a)-a$$

$$\Leftrightarrow y(x)=\frac{y(a)}{a}\,x \vee y(x)=y(a)+x-a$$

and the last two equations are reproductive. Hence, all the solutions of the equation

$$(y-xy')\,(y'-1)=0$$

are given by

$$y\,(x)=\frac{F(a)}{a}\,x \vee y\,(x)=F\,(a)+x-a$$

where $F \in D_1$ is an arbitrary function, or

$$y=Cx \vee y=C+x,$$

where C is arbitrary constant $\left(\text{in the first case } C=\frac{F(a)}{a}, \text{ and in the second } C=F(a)-a\right)$.

4. In paper [6] MAŽIROS speaks of differential equations having „several general solutions". By that he means two or more geometrically different families of integral curves. For instance, for the equation $(y-xy')(y'-1)=0$ from Example 3.3, MAŽIROS would say that it has two general solutions. In view of the ideas exposed here in Sections 2 and 3, such an approach seems to be incorrect.

We also note that the examples given in [6] are not well chosen.

EXAMPLE 4.1. For the equation $y'\,y'''-(y'')^2=0$ it is said in [6] that it has two different families of solutions, namely

$$y=ax+b \text{ and } y=\frac{a}{b}\,e^{bx}+c \qquad (a,\, b,\, c \text{ arbitrary constants}).$$

However, both families are contained in the expression

$$y=C+\int Ae^{Bx}\,dx \qquad (A,\, B,\, C \text{ arbitrary constants}).$$

EXAMPLE 4.2. For the equation

$$y'''(1+(y')^2)-2\,y'\,y''=0 \tag{4.1}$$

MAŽIROS states that it has the following different families of solutions

$$y=ax+b \text{ and } x^2+y^2+ax+by+c=0 \qquad (a,\, b,\, c, \text{ arbitrary constants}) \tag{4.2}$$

This is not correct — the eq. (4.1) should read

$$y'''\,(1+(y')^2)-3\,y'\,(y'')^2=0,$$

but independently from that, it is clear that the both families (4.2) are contained in

$$A\,(x^2+y^2)+Bx+Cy+D=0 \quad (A,\, B,\, C,\, D \text{ arbitrary constants}).$$

5. For a rather general class of nonlinear first order equations we can, in a way, identify the concepts „general solution" and „solution with an arbitrary constant".

In this case the term „general solution" is taken in a wider sense – not as a solution which *contains* all the solutions, but as a solution from which it is possible to *obtain* all the solutions.

Suppose that the equation

$$y'=F\,(x,\, y) \tag{5.1}$$

has the following solution with an arbitrary constant

(5.2) $$R(x, y, C)=0 \quad \left(C \text{ arbitrary constant; } \frac{\partial R}{\partial y}\neq 0\right).$$

If we introduce a new unknown function u by means of

(5.3) $$R(x, y, u)=0,$$

then (5.1) is reduced to $\frac{\partial R}{\partial u}u'=0$, i.e. to two equations

(5.4) $$u'=0 \text{ and } \frac{\partial R(x, y, u)}{\partial u}=0.$$

The first equation (5.4) implies $u=C$ (C arbitrary constant) which together with (5.3) gives the known solution (5.2).

The second equation (5.4) together with (5.3) gives parametric equations for the solutions not contained in (5.2).

In certain cases it is possible to apply this method to first order equations which are not explicitely solved with respect to y'.

REMARK. The procedure indicated above is a variant of the method of variation of constants.

EXAMPLE 5.1. The equation

(5.5) $$y' y^2+(xy'+y)^2=0$$

has the solution with an arbitrary constant

(5.6) $$y=\frac{C^2}{C+x} \quad (C \text{ arbitrary constant}).$$

Introduce the new function u by means of

(5.7) $$y=\frac{u^2}{u+x}.$$

Then, substituting (5.7), together with the corresponding expression for y', into (5.5), after some calculations, we find

$$u' u^2 (x^2 u'+u^2)(u+2x)^2=0.$$

(i) If $u'=0$, then $u=C$, which together with (5.7) gives the known solution (5.6).

(ii) If $u=0$, then from (5.7) follows $y=0$, and this solution is contained in (5.6), for $C=0$.

(iii) If $x^2 u'+u^2=0$, then $u=\frac{x}{Ax-1}$ (A arbitrary constant), and this combined with (5.7) yields

$$y=\frac{1}{A^2x-A}.$$

This solution, however, is contained in (5.6), and can be obtained from (5.6) for $C=-\frac{1}{A}$.

(iv) If $u+2x=0$, then from (5.7), we get $y=-4x$, which is a solution of (5.5) not contained in (5.6), i.e. the „singular solution".

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NEKE PRIMEDBE O OPŠTEM REŠENJU OBIČNIH DIFERENCIJALNIH JEDNAČINA

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U radu se raspravlja o pojmu opšteg rešenja običnih diferencijalnih jednačina koje se definiše kao rešenje koje obuhvata sva rešenja posmatrane jednačine. Pokazuje se, između ostalog, da je opšte rešenje definisano određenim preslikavanjem i jednom proizvoljnom funkcijom, bez obzira na red jednačine.

Opisana je i jedna klasa jednačina prvog reda kod koje se iz rešenja sa jednom proizvoljnom konstantom (koje ne mora biti opšte rešenje) mogu, na određen način, dobiti sva rešenja.