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723. O-REGULARLY VARYING FUNCTIONS*

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1. Introduction. In their paper [1], S. ALJANČIĆ and D. ARANĐELOVIĆ present an extensive study of O-regularly varying functions, short *O-RV functions*, i. e. of finite and measurable functions R, defined on $J_a=[a, \infty)$ for some $a>0$, such that

$$\limsup_{x\to\infty} \frac{R(tx)}{R(x)} = r(t) \tag{1.1}$$

is positive and finite for each $t>0$.

The paper [1] is restricted just to the study of O-RV functions. The paper [2] of R. BOJANIĆ and M. VUILLEUMIER gives a sample of theorems on regular operators (and, a fortiori, on integral transformations) which involve O-RV functions.

O-RV functions were introduced by V. G. AVAKUMOVIĆ in [3] and extensively studied by J. KARAMATA in [4] and several other papers. Independently of this research N. K. BARI and S. B. STEČKIN have considered a subfamily of the same class of functions in [5], their well-known memoir on best approximation.

The aim of this paper is an investigation into the nature of the O-RV functions and of what makes them really appropriate for the study of the asymptotic behavior of integral transforms. Roughly speaking, O-RV functions behave like the powers $g_\alpha(x)=x^\alpha$, $\alpha>0$; this loose characterization is sharpened in this paper through a relativization of the notion of an O-RV function, which shows better their true nature. Then this relativization is used to establish some theorems on the O-behavior of regular operators, which contain the basic theorem of [2] as a very special case.

2. Relativization. Let $\mathcal{F}$ denote the vector space of all measurable functions $f:\mathbf{R}^+\to\mathbf{R}$, where $\mathbf{R}$ is the set of all reals and $\mathbf{R}^+$ the set of all nonnegative reals; let $\mathcal{M}\subset\mathcal{F}$ be the vector subspace of $\mathcal{F}$ composed of all locally bounded elements of $\mathcal{F}$ (f is *locally bounded* if it is bounded on every finite interval of $\mathbf{R}^+$), and let $\mathcal{B}\subset\mathcal{M}$ denote the Banach space of all bounded elements f of $\mathcal{F}$, vith the norm $\|f\|=\sup_{x\in\mathbf{R}^+}|f(x)|$.

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A function $g \in \mathcal{M}$, which is positive, increasing and $\geqq 1$ outside of the interval [0, 1], will be called a *basic function.*

Definition 2.1. (*a*). *Let g be a basic function. A positive $L \in \mathcal{M}$ is called a gRV-function if there are positive numbers M_1 aud M_2 such that for all $t \geqq 0$*

$$L(t)g(t) \leqq M_1 L(x) g(x) \text{ for } t \in [0, x], \tag{2.1}$$

and

$$L(t)g(x) \leqq M_2 L(x) g(t) \text{ for } t \in [x, \infty). \tag{2.2}$$

(*b*). *If G is a family of basic functions we say that L is a GRV-function if it is a gRV-function for some $g \in G$.*

For example, if $0 < m < M$ and $mg(t) \leqq L(t) \leqq Mg(t)$ for $t \geqq 0$, L is a gRV-function. Similarly, if $g_\alpha(x) = x^\alpha$, $\alpha > 0$, and $G = \{g_\alpha \mid \alpha > 0\}$, the class of all GRV-functions is just the class of all O-RV functions. To see this remark that (2.1) and (2.2) with $g = g_\alpha$ give

$$\frac{1}{M_1}\left(\frac{t}{x}\right)^\alpha \leqq \frac{L(x)}{L(t)} \leqq M_2 \left(\frac{x}{t}\right)^\alpha \text{ for } x > t,$$

from where follows easily

$$\frac{1}{M_1} \cdot \lambda^{-\alpha} \leqq \frac{L(tx)}{L(x)} \leqq M_2 \lambda^\alpha \text{ for } 1 \leqq t \leqq \lambda. \tag{2.3}$$

But (2.3) falls under the characterization of the O-RV functions given by KARAMATA in [4] (See also [1], footnote 2 on page 7).

In general, (2.1) and (2.2) imply the existence of positive numbers m and M such that, for all $\lambda \geqq 1$ and $x \in \mathbf{R}^+$

$$m \frac{g(x)}{g(\lambda x)} \leqq \frac{L(\lambda x)}{L(x)} \leqq M \frac{g(\lambda x)}{g(x)}. \tag{2.4}$$

Another immediate consequence of (2.1) and (2.2) are the relations (for $x \to \infty$)

$$\sup_{t \in [0, x]} g(t) L(t) = O\big(g(x) L(x)\big) \tag{2.5}$$

and

$$\sup_{t \in [x, \infty)} \frac{L(t)}{g(t)} = O\left(\frac{L(x)}{g(x)}\right). \tag{2.6}$$

As it will be shown in the next section, (2.5) and (2.6) are the only relations important in the O-behavior of linear transforms involving a gRV-function L.

In this paper we will not enter into an extensive study of the gRV-functions, say on the lines of the paper [1]. Our aim is to exhibit their role in the O-behavior of linear operations.

3. Regular Operators and the g RV-Functions. We shall consider functionals $\psi : \mathcal{X} \times \mathbf{R}^+ \to \mathbf{R}$, where $\mathcal{X} \subset \mathcal{F}$ is a vector lattice which contains $\mathcal{B}$. (Thus, with a functions f, $\mathcal{X}$ contains its positive and its negative parts, and with two functions f and h, $\mathcal{X}$ contains their upper and their lower envelope). The functional ψ above is called *linear* if it is linear in its first argument, i.e. if

$$\psi(af+bh, x) = a\psi(f, x) + b\psi(h, x), \tag{3.1}$$

for all $a, b \in \mathbf{R}$ and $f, h \in \mathcal{X}$.

Positive linear functionals ψ satisfy, for all $f \in \mathcal{X}$,

$$|\psi(f, x)| \leqq \psi(|f|, x), \text{ for all } x \in \mathbf{R}^+, \tag{3.2}$$

and, for all $f \in \mathcal{B}$,

$$|\psi(f, x)| \leqq \psi(1, x)\,\|f\|, \text{ for all } x \in \mathbf{R}^+, \tag{3.3}$$

where 1 denotes the function taking the value one for all the values of its argument.

Every functional $\psi : \mathcal{X} \times \mathbf{R}^+ \to \mathbf{R}$ induces an operator $\bar{\psi}$ in a unique way: $\bar{\psi}(f) = h$ if $h(x) = \psi(f, x)$ for all $x \in \mathbf{R}^+$. In general, we shall use this identification and call ψ also an operator.

An operator Φ is called *regular* if it is the difference of two positive linear operators. The bahavior of such an operator depends essentially on the functional V_Φ, defined for every nonnegative, masurable $f: \mathbf{R}^+ \to \mathbf{R}^+$ by

$$V_\Phi(f, x) = \sup\{|\Phi(h, x)| \mid h \in \mathcal{X}, |h| \leqq f\}. \tag{3.4}$$

Let $g_\alpha(x) = x^\alpha$, $\alpha > 0$, and let H_A denote the characteristic function of the set A (i.e. $H_A(x) = 1$ iff $x \in A$, and $H_A(x) = 0$ iff $x \notin A$). The basic theorem 4 of the section 2.3 of [2] is then:

Theorem 3.1. *Let Φ be a regular operator on $\mathcal{M}$. In order that*

$$\Phi(L, x) = O(L(x)),\ x \to \infty, \tag{3.5}$$

holds for every O-RV function $L \in \mathcal{M}$, it is necessary and sufficient that, for every $\alpha > 0$,

$$V_\Phi(g_\alpha, x) = O(x^\alpha),\ x \to \infty, \tag{3.6}$$

and

$$V_\Phi(H_{[0,1]} + H_{(1,\infty)} \cdot (1/g_\alpha), x) = O(x^{-\alpha}),\ x \to \infty. \tag{3.7}$$

The main theorem of this paper is a relativized version of the theorem 3.1. This is

Theorem 3.2. *Let $\Phi : \mathcal{X} \to \mathcal{F}$ be a regular operator and let $g \in \mathcal{X}$ be a basic function. Then, for every gRV-function $L \in \mathcal{X}$,*

$$\Phi(L, x) = O(L(x)),\ x \to \infty, \tag{3.8}$$

if and only if

$$V_\Phi(g, x) = O(g(x)),\ x \to \infty, \tag{3.9}$$

and

$$V_\Phi(s_g, x) = O(1/g(x)),\ x \to \infty, \tag{3.10}$$

where

$$s_g(t) = H_{[0,1]}(t) + H_{(1,\infty)}(t)/g(t). \tag{3.11}$$

Proof. Let $L \in \mathcal{X}$ be a *gRV*-function. Then, by linearity of Φ, for every $x \in \mathbf{R}^+$,

$$|\Phi(L, x)| \leqq |\Phi(L \cdot H_{[0, x]}, x)| + |\Phi(L \cdot H_{(x, \infty)}, x)|. \tag{3.12}$$

Remark now that

$$L(t) \cdot H_{[0, x]}(t) = \{L(t)[H_{[0, 1]}(t) + g(t) \cdot H_{(1, \infty)}(t)] \cdot H_{[0, x]}(t)\} \cdot s_g(t),$$

and that, for $t \in [1, \infty)$,

$$L(t) \cdot H_{[1, \infty)}(t) = g(t) \cdot \left\{\frac{L(t)}{g(t)}\right\} \cdot H_{(1, \infty)}(t).$$

Thus, if we introduce the functions

$$s(x) = \sup_{t \in [0, x]} \{L(t) \cdot [H_{[0, 1]}(t) + g(t) \cdot H_{(1, \infty)}(t)]\}$$

and

$$S(x) = \sup_{t \in [x, \infty)} \{L(t)/g(t)\},$$

we have

$$L(t) \cdot H_{[0, x]}(t) \leqq s(x)\, s_g(t) \tag{3.13}$$

and

$$L(t) \cdot H_{(x, \infty)}(t) \leqq S(x)\, g(t). \tag{3.14}$$

With this, from (3.12) follows easily

$$\frac{1}{L(x)} |\Phi(L, x)| \leqq \frac{s(x)}{g(x) \cdot L(x)} \cdot g(x) \cdot V_\Phi(s_g, x) + \frac{S(x)}{L(x)/g(x)} \cdot \frac{1}{g(x)} \cdot V_\Phi(g, x),$$

from where, using (2.5), (2.6) and the suppositions of the theorem, (3.8) follows immediately. This proves that (3.9) and (3.10) are sufficient conditions. That they are necessary can be proved similarly as in the proof of theorem 4 of [2], using g instead of x^α and s_g instead of the function introduced at the end of that proof.

From theorem 3.2. follows immediately

Theorem 3.3. *Let $\Phi : \mathcal{X} \to \mathcal{F}$ be a regular operator and G a family of basic functions. In order that, for every GRV-function $L \in \mathcal{X}$,* (3.8) *holds it is necessary and sufficient that* (3.9) *and* (3.10) *hold for every $g \in G$.*

Theorem 3.3 contains theorem 3.1 (i.e. theorem 4 of [2]) as a very special case, taking $\mathcal{X} = \mathcal{M}$ and $G = \{g_\alpha \mid \alpha > 0\}$, where $g_\alpha(x) = x^\alpha$.

Obviously, theorem 3.2 does not exhaust all the possibilities for application of the *gRV*-functions. One could pursue the subject further, in the spirit of [2]. We shall reserve this pursuit for another paper. Here, our aim was just to uncover the structure of the *O-RV* functions which makes possible their use in the study of the asymptotic behavior of operators. Some results will be published in [6].

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O O-REGULARNO PROMENLJIVIN FUNKCIJAMA

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Takozvane *O*-regularne funkcije predstavljaju graničnu klasu funkcija čije asimptotsko ponašanje uslovljava regularno ponašanje njihovih integralnih transformacija.

Cilj autora je da karakteriše osnovne osobine *O*-regularno promenljivih funkcija koje omogućuju takvo ponašanje. U tu svrhu on *relativizira* pojam *O*-regularne promenljivosti i uspostavlja osnovne relacije te relativizacije uslovima (2.1) i (2.2). Na taj način dolazi se do teoreme 3.2 koja, kao specijalne slučajeve, sadrži skoro sve stavove o ponašanju integralnih transformacija *O*-regularno promenljivih funkcija.