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722. ON A CLASS OF CYCLIC FUNCTIONAL EQUATIONS*

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0. In this paper we shall give continuous real solutions of some nonlinear cyclic functional equations, i.e. the functions F, f_i $(i=1, \dots, m+n)$ are real and continuous functions of two variable.

1. The general solution of the functional equation (see [1—4]):

$$\sum_{i=1}^{m+n} C^{i-1} f_i(x_1+\cdots+x_m,\ x_{m+1}+\cdots+x_{m+n})=0 \tag{1}$$

where the cyclic operator C is given by

$$Cf(x_1,\ x_2, \dots, x_{m+n})=f(x_2, \dots, x_{m+n},\ x_1), \tag{2}$$

is

$$f_i(x, y)=a(x+y)(nx-my)+c_i(x+y) \qquad (1\leqq i\leqq m+n), \tag{3}$$

where a, $c_1, \dots, c_{m+n}$ are arbitrary functions such that

$$\sum_{i=1}^{m+n} c_i(x)=0. \tag{4}$$

REMARK. 1° The following results are variants of the functional equation (1):

a) The functinal equation

$$\sum_{i=1}^{m+n} C^{i-1} f_i(x_1\cdots x_m,\ x_{m+1}\cdots x_{m+n})=0 \qquad (x_i>0\ i=1, \dots, m+n) \tag{5}$$

has a general solution

$$f_i(x,\ y)=a(xy)\log\frac{x^n}{y^m}+c_i(xy) \qquad (1\leqq i\leqq m+n) \tag{6}$$

where a, $c_1, \dots, c_{m+n}$ are arbitrary functions such that (4) holds.

b) The general solution of the functional equation

$$\prod_{i=1}^{m+n} C^{i-1} f_i(x_1+\cdots+x_m,\ x_{m+1}+\cdots+x_{m+n})=1 \tag{7}$$

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is

(8) $$f_i(x, y) = c_i(x+y)\, e^{a(x+y)(nx-my)} \qquad (1 \le i \le m+n)$$

where a, $c_1>0, \ldots, c_{m+n}>0$ are arbitrary functions such that

(9) $$\prod_{i=1}^{m+n} c_i(x) = 1.$$

c) The general solution of the functional equation

(10) $$\prod_{i=1}^{m+n} C^{i-1} f_i(x_1 \cdots x_m, x_{m+1} \cdots x_{m+n}) = 1 \qquad (x_i > 0,\ 1 \le i \le m+n)$$

is

(11) $$f_i(x, y) = c_i(xy) \left(\frac{x^n}{y^m}\right)^{a(xy)} \qquad (1 \le i \le m+n).$$

where a, $c_1>0, \ldots, c_{m+n}>0$ are arbitrary functions subject to (9).

1.1. We shall denote by $E_k(x_1, \ldots, x_n)$ the k-th elementary symmetric function of $x_1, \ldots, x_n$, i.e.,

$$E_k(x_1, \ldots, x_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Let α be a real number and let

$$I_n(x_1, \ldots, x_n; \alpha) = \sum_{k=1}^{n} \alpha^{k-1} E_k(x_1, \ldots, x_n).$$

The general solution of the functional equation

(12) $$\sum_{i=1}^{m+n} C^{i-1} f_i(I_m(x_1, \ldots, x_m; \alpha), I_n(x_{m+1}, \ldots, x_{m+n}; \alpha)) = 0$$

is

(13) $$f_i(x, y) = b((\alpha x+1)(\alpha y+1)) \log \frac{(\alpha x+1)^n}{(\alpha y+1)^m} + c_i((\alpha x+1)(\alpha y+1))$$
$$(1 \le i \le m+n)$$

where $b, c_1, \ldots, c_{m+n}$ are arbitrary functions such that (4) holds.

Since

$$I_n(x_1, \ldots, x_n; \alpha) = \frac{1}{\alpha}\left(\prod_{k=1}^{n} (\alpha x_k + 1) - 1\right)$$

using the substitutions

$$\alpha x_i + 1 = u_i \text{ and } f_i\left(\frac{1}{\alpha}(u-1), \frac{1}{\alpha}(v-1)\right) = F(u, v) \qquad (1 \le i \le m+n),$$

(12) becomes

$$\sum_{i=1}^{m+n} C^{i-1} F_i(u_1 \cdots u_m,\ u_{m+1} \cdots u_{m+n}) = 0.$$

Using (6), we obtain

$$F_i(u,\ v) = b(uv) \log \frac{u^n}{v^m} + c_i(uv),$$

where $b, c_1, \ldots, c_{m+n}$ are arbitrary functions such that (4) holds. So,

$$f_i\left(\frac{1}{\alpha}(u-1),\ \frac{1}{\alpha}(v-1)\right) = b(uv) \log \frac{u^n}{v^m} + c_i(uv)$$

i.e. $f_i(x, y)$ is given by (13).

Similarly, we can get the following results:

The general solution of the functional equation

$$\prod_{i=1}^{m+n} C^{i-1} f_i(I_m(x_1, \ldots, x_m;\ \alpha),\ I_n(x_{m+1}, \ldots, x_{m+n};\ \alpha)) = 1 \tag{14}$$

$(x_i > -1/\alpha,\ \ 1 \leqq i \leqq m+n)$, is

$$f_i(x, y) = c_i((\alpha x+1)(\alpha y+1)) \left(\frac{(\alpha x+1)^n}{(\alpha y+1)^m}\right)^{b((\alpha x+1)(\alpha y+1))} \quad (1 \leqq i \leqq m+n), \tag{15}$$

where $b, c_1 > 0, \ldots, c_{m+n} > 0$ are arbitrary functions subject to (9).

The general solution of the functional equation

$$I_{m+n}(F_1, \ldots, F_{m+n};\ \alpha) = 0 \qquad (\alpha \in \mathbf{R}), \tag{16}$$

where

$$F_i = C^{i-1} f_i(x_1 + \cdots + x_m,\ x_{m+1} + \cdots + x_{m+n}) \qquad (1 \leqq i \leqq m+n) \tag{17}$$

is

$$f_i(x,\ y) = \frac{1}{\alpha}(c_i(x+y)\, e^{b(x+y)(nx-my)} - 1) \qquad (1 \leqq i \leqq m+n), \tag{18}$$

where $b, c_1 > 0, \ldots, c_{m+n} > 0$ are arbitrary functions such that (9) is valid.

The general solution of the functional equation (16) where

$$F_i = C^{i-1} f_i(x_1 \cdots x_m,\ x_{m+1} \cdots x_{m+n}) \qquad (1 \leqq i \leqq m+n) \tag{19}$$

is

$$f_i(x, y) = \frac{1}{\alpha}\left(c_i(xy)\left(\frac{x^n}{y^m}\right)^{a(xy)} - 1\right) \qquad (1 \leqq i \leqq m+n), \tag{20}$$

where $a,\ c_1 > 0, \ldots, c_{m+n} > 0$ are arbitrary functions subject to (9).

The general solution of the functional equaton (16), where

(21) $$F_i = C^{i-1} f_i(I_m(x_1, \dots, x_m; \alpha), I_n(x_{m+1}, \dots, x_{m+n}; \alpha))$$

$(x_i > -1/\alpha,\ 1 \leq i \leq m+n)$, is

(22) $$f_i(x, y) = \frac{1}{\alpha}\left(c_i((\alpha x+1)(\alpha y+1))\left(\frac{(\alpha x+1)^n}{(\alpha y+1)^m}\right)^{b((\alpha x+1)(\alpha y+1))} - 1\right)$$
$$(1 \leq i \leq m+n),$$

where b, $c_1 > 0, \dots, c_{m+n} > 0$ are arbitrary functions such that (9) is valid.

1.2. We shall write

$$J_n(x_1, \dots, x_n; \alpha) = \sum_{k=1}^{n} k\alpha^{k-1} E_k(x_1, \dots, x_n).$$

The general solution of the functional equation

(23) $$J_{m+n}(F_1, \dots, F_{m+n}; \alpha) = 0 \qquad (\alpha \in \mathbf{R}),$$

where $F_i (1 \leq i \leq m+n)$ are given by (17), is

(24) $$f_i(x, y) = -\frac{1}{\alpha}\,\frac{b(x+y)(nx-my)+c_i(x+y)}{b(x+y)(nx-my)+c_i(x+y)-1} \qquad (1 \leq i \leq m+n),$$

where b, $c_1, \dots, c_{m+n}$ are arbitrary functions such that (4) holds; or

(25) $$f_i(x, y) = -1/\alpha, \quad f_j(x, y) = -1/\alpha \ (i \neq j),$$
$$f_k(x, y)\ (1 \leq k \leq m \leq n,\ k \neq i, j) \text{ are arbitrary functions.}$$

Indeed, by substitutions $f_i(x, y) \to -\frac{1}{\alpha}\frac{f_i(x, y)}{f_i(x, y)-1} (1 \leq i \leq m+n)$, and using the identity

$$J_n\left(-\frac{1}{\alpha}\frac{y_1}{y_1-1}, \dots, -\frac{1}{\alpha}\frac{y_n}{y_n-1}; \alpha\right) = -\frac{1}{\alpha}\frac{y_1+\dots+y_n}{(y_1-1)\cdots(y_n-1)},$$

we can get (24). Now, let $f_i(x, y) = -1/\alpha$. Then (23) becomes

$$\prod_{\substack{j=1\\ j\neq i}}^{m+n} (\alpha F_j + 1) = 0,$$

wherefrom we have (25).

Analogously, we can get the following results:

The general solution of the functional equation (23), where F_i $(1 \leq i \leq m+n)$ are given by (19), is

(26) $$f_i(x,y) = -\frac{1}{\alpha}\,\frac{b(xy)\log(x^n/y^m)+c_i(xy)}{b(xy)\log(x^n/y^m)+c_i(xy)-1} \qquad (1 \leq i \leq m+n),$$

or (25), where b, $c_1, \dots, c_{m+n}$ are arbitrary functions such that (4) holds.

1.3. Now, let

$$S_n(x_1, \dots, x_n; \alpha, \varepsilon) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \varepsilon^k \alpha^{2k} E_{2k+1}(x_1, \dots, x_n).$$

The general solution of the functional equation

$$\text{(27)} \qquad S_{m+n}(F_1, \dots, F_{m+n}; \alpha, \varepsilon) = 0 \qquad (\alpha > 0),$$

where F_i $(1 \leqq i \leqq m+n)$ are given by (17), is

a) for $\varepsilon = -1$,

$$\text{(28)} \qquad f_i(x, y) = \frac{1}{\alpha} \operatorname{tg}(b(x+y)(nx-mn) + c_i(x+y)) \qquad (1 \leqq i \leqq m+n),$$

where $b, c_1, \dots, c_{m+n}$ are arbitrary functions such that (4) is valid;

b) for $\varepsilon = 1$,

$$\text{(29)} \qquad f_i(x, y) = \frac{1}{\alpha} \operatorname{th}(b(x+y)(nx-my) + c_i(x+y)) \qquad (1 \leqq i \leqq m+n),$$

where $b, c_1, \dots, c_{m+n}$ are arbitraty functions such that (4) is valid, or

$$\text{(30)} \qquad \begin{gathered} f_i(x, y) = 1/\alpha, \quad f_j(x, y) = -1/\alpha \ (i \neq j), \\ f_k(x, y) \ (1 \leqq k \leqq m+n, \ k \neq i, j) \ \text{ are arbitrary functions.} \end{gathered}$$

This result is a generalization of functional equation from [6] (see also [7]).

Indeed, using the substitutions $F_i \to \operatorname{tg} F_i$ $(\operatorname{th} F_i)$ $(1 \leqq i \leqq m+n)$, we have that equation (27) is equivalent to

$$\operatorname{tg}(F_1 + \cdots + F_{m+n}) = 0 \ \ (\text{or } \operatorname{th}(F_1 + \cdots + F_{m+n}) = 0)$$

wherefrom we can get (28) (or (29)).

Now, let $f_i(x, y) = 1/\alpha$. Then (27) for $\varepsilon = 1$, becomes $\prod_{\substack{k=1 \\ k \neq i}}^{m+n} (1 + \alpha F_k) = 0$, wherefrom we have (30).

Analogously, we can obtain the following result:

The general solution of the functional equation (27), where F_i $(1 \leqq i \leqq m+n)$ are given by (19), is

a) for $\varepsilon = -1$,

$$\text{(31)} \qquad f_i(x, y) = \frac{1}{\alpha} \operatorname{tg}(b(xy) \log(x^n/y^m) + c_i(xy)) \qquad (1 \leqq i \leqq m+n)$$

where $b, c_1, \dots, c_{m+n}$ are arbitrary functions such that (4) is valid;

b) for $\varepsilon = 1$,

$$\text{(32)} \qquad f_i(x, y) = \frac{1}{\alpha} \operatorname{th}(b(xy) \log(x^n/y^m) + c_i(xy)) \qquad (1 \leqq i \leqq m+n)$$

or (30), where b, $c_1, \ldots, c_{m+n}$ are arbitrary functions such that (4) is valid.

REMARK. 2° Using the ideas from [5], we can get some other analogous results.

2. The general solution of the functional equation (see [3]):

$$(33)\quad \sum_{i=1}^{n+m+p} C^{i-1} F(x_1+\cdots+x_m, x_{m+1}+\cdots+x_{m+n})=0 \quad (m, n, p\in \mathbf{N})$$

where the cyclic operator C is given by

$$(34)\quad Cf(x_1, x_2, \ldots, x_{m+n})=f(x_2, \ldots, x_{m+n}, x_{m+n+1}) \quad (x_{m+n+p+i}=x_i),$$

is

$$(35)\quad F(x, y)=c(nx-my) \quad (n\neq m),$$

$$(36)\quad F(x, y)=h(x)-h(y) \quad (n=m),$$

where, here and in the next text, c is an arbitrary constant and h is an arbitrary function.

REMARK. 3° The following results are variants of the functional equation (33).

a) The general solution of the functional equation

$$(37)\quad \sum_{i=1}^{m+n+p} C^{i-1} F(x_1\cdots x_m, x_{m+1}\cdots x_{m+n})=0 \quad (x_i>0,\ 1\leqq i\leqq m+n+p)$$

is

$$(38)\quad \begin{aligned} F(x, y)&=c\log(x^n/y^m) && (n\neq m),\\ F(x, y)&=h(x)-h(y) && (n=m). \end{aligned}$$

b) The general solution of the functional equation

$$(39)\quad \prod_{i=1}^{m+n+p} C^{i-1} F(x_1+\cdots+x_m, x_{m+1}+\cdots+x_{m+n})=1$$

is

$$(40)\quad F(x, y)=e^{c(nx-my)} \quad (n\neq m),$$

$$(41)\quad F(x, y)=h(x)/h(y) \quad (n=m,\ h \text{ is positive function}).$$

c) The general solution of the functional equation

$$(42)\quad \prod_{i=1}^{m+n+p} C^{i-1} F(x_1\cdots x_m, x_{m+1}\cdots x_{m+n})=1 \quad (x_i>0,\ 1\leqq i\leqq m+n+p)$$

is

$$(43)\quad F(x, y)=(x^n/y^m)^c \quad (n\neq m), \text{ and (41) for } n=m.$$

2.1. The general solution of the functional equation

$$(44)\quad \sum_{i=1}^{m+n+p} C^{i-1} F(I_m(x_1, \ldots, x_m; \alpha), I_n(x_{m+1}, \ldots, x_{m+n}; \alpha))=0 \quad (\alpha\in \mathbf{R}),$$

where I_n is defined as in **1.1**, $x_i > -1/\alpha$ for $1 \leqq i \leqq m+n+p$, is

(45) $$F(x, y) = c \log \frac{(\alpha x+1)^n}{(\alpha y+1)^m} \quad (n \neq m), \text{ and (38) for } n=m.$$

The general solution of the functional equation

(46) $$\prod_{i=1}^{m+n+p} C^{i-1} F(I_m(x_1, \dots, x_m; \alpha), I_n(x_{m+1}, \dots, x_{m+n}; \alpha)) = 1$$

$(x_i > -1/\alpha, \ 1 \leqq i \leqq m+n+p)$, is

(47) $$F(x, y) = \frac{(\alpha x+1)^{cn}}{(\alpha y+1)^{cm}} (n \neq m), \text{ and (41) for } n=m.$$

The general solution of the functional equation

(48) $$I_{m+n+p}(F_1, \dots, F_{m+n+p}; \alpha) = 0 \qquad (\alpha \in \mathbf{R}),$$

where

(49) $$F_i = C^{i-1} F(x_1 + \dots + x_m, x_{m+1} + \dots + x_{m+n}) \qquad (1 \leqq i \leqq m+n+p),$$

is

(50) $$F(x, y) = \frac{1}{\alpha} (e^{c(nx-my)} - 1) \qquad (n \neq m),$$

(51) $$F(x, y) = \frac{1}{\alpha} (h(x)/h(y) - 1) \qquad (n=m, \ h \text{ is positive function}).$$

The general solution of the functional equation (48), where

(52) $$F_i = C^{i-1} F(x_1 \cdots x_m, x_{m+1} \cdots x_{m+n}) \ (1 \leqq i \leqq m+n+p),$$

$(x_i > 0, \ 1 \leqq i \leqq m+n+p)$, is

(53) $$F(x, y) = \frac{1}{\alpha} ((x^n/y^m)^c - 1) \ (n \neq m), \text{ and (51) for } n=m.$$

The general solution of the functional equation (48), where

(54) $$F_i = C^{i-1} F(I_m(x_1, \dots, x_m; \alpha), I_n(x_{m+1}, \dots, x_{m+n}; \alpha))$$

$(x_i > -1/\alpha, \ (1 \leqq i \leqq m+n+p)$, is

(55) $$F(x, y) = \frac{1}{\alpha} \left(\frac{(\alpha x+1)^{cn}}{(\alpha y+1)^{cm}} - 1 \right) \ (n \neq m), \text{ and (51) for } n=m.$$

2.2. The general solution of the functional equation

(56) $$J_{n+m+p}(F_1, \dots, F_{m+n+p}; \alpha) = 0 \ (\alpha \in \mathbf{R}),$$

where J_n is defined as in **1.2.** and F_i $(1 \leqq i \leqq m+n+p)$ are given by (49), is

(57) $$F(x, y) = -\frac{1}{\alpha} \frac{c(nx-my)}{c(nx-my)-1} \qquad (n \neq m),$$

(58) $$F(x, y) = -\frac{1}{\alpha} \frac{h(x)-h(y)}{h(x)-h(y)-1} \qquad (n=m),$$

or

(59) $$F(x, y) = -1/\alpha.$$

The general solution of the functional equation (56), where F_i $(1 \leqq i \leqq m+n+p)$ are given by (52), is

(60) $$F(x, y) = -\frac{1}{\alpha}\frac{c\log(x^n/y^m)}{c\log(x^n/y^m)-1} \quad (n \neq m), \text{ and (58) for } n=m;$$

or (59).

2.3. The general solution of the functional equation

(61) $$S_{m+n+p}(F_1, \ldots, F_{m+n+p}; \alpha, \varepsilon) = 0 \qquad (\alpha > 0),$$

where S_n is defined as in **1.3.** and F_i $(1 \leqq i \leqq m+n+p)$ are given by (49), is:

a) for $\varepsilon = -1$,

(62) $$F(x, y) = \frac{1}{\alpha}\operatorname{tg}(cnx - cmy) \qquad (n \neq m),$$

(63) $$F(x, y) = \frac{1}{\alpha}\operatorname{tg}(h(x) - h(y)) \qquad (n = m);$$

b) for $\varepsilon = 1$,

(64) $$F(x, y) = \frac{1}{\alpha}\operatorname{th}(cnx - cmy) \qquad (n \neq m),$$

(65) $$F(x, y) = \frac{1}{\alpha}\operatorname{th}(h(x) - h(y)) \qquad (n = m).$$

The general solution of the functional equation (61), where F_i $(1 \leqq i \leqq m+n+p)$ are given by (52), is:

a) for $\varepsilon = -1$,

(66) $$F(x, y) = \frac{1}{\alpha}\operatorname{tg}\left(c\log\frac{x^n}{y^m}\right) \quad (n \neq m), \text{ and (63) for } n=m,$$

b) for $\varepsilon = 1$,

(67) $$F(x, y) = \frac{1}{\alpha}\operatorname{th}\left(c\log\frac{x^n}{y^m}\right) \quad (n \neq m), \text{ and (65) for } n=m.$$

REMARKS. 4° The functional equation

$$\sum_{i=1}^{m+n+p} C^{i-1} F_i(x_1 + \cdots + x_m, x_{m+1} + \cdots + x_{m+n}) = 0$$

is also considered in [3], and the solution for $m=2$, $n=1$, $p=1$, is given. Using this result, we can get some results which are analogous to the previous results.

5° We can get some analogous results for equation (III.3.1) from [3, p. 39], too.

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JEDNA KLASA CIKLIČNIH FUNKCIONALNIH JEDNAČINA

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Koristeći rešenja linearnih cikličnih funkcionalnih jednačina (1) i (33), dobijena su rešenja za niz nelinearnih cikličnih funkcionalnih jednačina. Neki od dobijenih rezultata predstavljaju generalizacije rezultata iz [6] i [7].