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720. ON AN INEQUALITY*

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If $a_1, \ldots, a_n$ $(n \geqq 2)$ are real numbers, then

$$\min_{1 \leqq i < k \leqq n} (a_k - a_i)^2 \leqq \frac{12}{n^2-1} \left[\frac{1}{n} \sum_{k=1}^{n} a_k^2 - \left(\frac{1}{n} \sum_{k=1}^{n} a_k \right)^2 \right]. \tag{1}$$

This inequality was established by S. B. PREŠIĆ ([1], p. 341, see also [2]) as an answer to a problem proposed by Professor D. S. MITRINOVIĆ in The American Mathematical Monthly.

Our aim is to prove some inequalities which are similar with (1). Let $w_1, \ldots, w_n$ be non-negative numbers such that $w_1 + \cdots + w_n = 1$. If $a = (a_1, \ldots a_n)$, $b = (b_1, \ldots, b_n)$ are systems of real numbers, then we use the following notation

$$m(a) = \min_{1 \leqq i < n} |a_{i+1} - a_i|, \quad M(a) = \max_{1 \leqq i < n} |a_{i+1} - a_i|,$$

$$D_n(w; a, b) = \sum_{k=1}^{n} w_k a_k b_k - \left(\sum_{k=1}^{n} w_k a_k \right) \left(\sum_{k=1}^{n} w_k b_k \right),$$

$$D_n(a, b) = \frac{1}{n} \sum_{k=1}^{n} a_k b_k - \left(\frac{1}{n} \sum_{k=1}^{n} a_k \right) \left(\frac{1}{n} \sum_{k=1}^{n} b_k \right),$$

$$e = (1, \ldots, n).$$

Observing that $D_n(a, a)$ does not depend on the order of summands, the inequality (1) may be written as

$$[m(a)]^2 \leqq \frac{D_n(a, a)}{D_n(e, e)}, \quad a_1 \leqq \cdots \leqq a_n.$$

Firstly we want to prove the following proposition

Theorem 1. *If* $a_1 \leqq \cdots \leqq a_n$, *then*

$$[m(a)]^2 \leqq \frac{12}{n^2-1} D_n(a, a) \leqq [M(a)]^2. \tag{2}$$

Moreover, the equality cases are attained if and only if

$$a_k = a_1 + (k-1)r, \quad k = 1, \ldots, n.$$

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Proof. We have

$$(3)\qquad D_n(a, a)=\frac{1}{n^2}\sum_{1\leqq i<j\leqq n}(a_j-a_i)^2=\frac{1}{n^2}\sum_{1\leqq i<j\leqq n}\left[\sum_{\nu=i}^{j-1}(a_{\nu+1}-a_\nu)\right]^2.$$

But, for $1\leqq i<j\leqq n$

$$(4)\qquad (j-i)\,m(a)\leqq\sum_{\nu=i}^{j-1}(a_{\nu+1}-a_\nu)\leqq(j-i)\,M(a)$$

with equality iff $a_2=a_1=a_3-a_2=\cdots=a_n-a_{n-1}$.
Now

$$\frac{1}{n^2}\sum_{1\leqq i<j\leqq n}(j-i)^2=D_n(e, e)=\frac{n^2-1}{12}.$$

Therefore (3) and (4) enables us to write the inequalities

$$[m(a)]^2 D_n(e, e)\leqq D_n(a, a)\leqq[M(a)]^2 D_n(e, e)$$

which complete the proof.

Theorem 2. *If* $a=(a_1, \ldots, a_n)$, $b=(b_1, \ldots, b_n)$ *are monotonic in the same sense, then*

$$(5)\qquad m(a)\,m(b)\leqq\frac{D_n(w;\, a, b)}{D_n(w;\, e, e)}\leqq M(a)\,M(b).$$

Moreover, if there are the numbers α, α_1, r, r_1 $(r\,r_1>0)$ *such that* $a_k=\alpha+kr$, $b_k=\alpha_1+kr_1$ *then in* (5) *hold the equqlity cases.*

Proof. According to the Binet-Cauchy identity one finds

$$(6)\qquad D_n(w;\, a, b)=\sum_{1\leqq i<j\leqq n}w_iw_j(a_j-a_i)(b_j-b_i)$$

$$=\sum_{1\leqq i<j\leqq n}w_i\,w_j\left[\sum_{\nu=i}^{j-1}(a_{\nu+1}-a_\nu)\right]\cdot\left[\sum_{\nu=i}^{j-1}(b_{\nu+1}-b_\nu)\right].$$

From (4) and (6) we conclude with (5).

If we select $w_1=\cdots=w_n$ then (5) is the same with

$$m(a)\,m(b)\leqq\frac{12}{n^2-1}\left[\frac{1}{n}\sum_{k=1}^{n}a_k\,b_k-\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n}b_k\right)\right]\leqq M(a)\,M(b),$$

where $a=(a_1, \ldots, a_n)$, $b=(b_1, \ldots, b_n)$ are „synchron", i. e. monotonic in the same sense.

The author is interested in the solutions of the following questions:
Problem 1. Which are the best constants x_n, y_n in the inequalities

$$x_n\cdot\min_{1\leqq i<j\leqq n}\begin{vmatrix}a_i & a_j\\ b_i & b_j\end{vmatrix}^2\leqq\left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k=1}^{n}b_k^2\right)-\left(\sum_{k=1}^{n}a_k\,b_k\right)^2$$

$$\leqq y_n\cdot\max_{1\leqq i\leqq j<n}\begin{vmatrix}a_i & a_j\\ b_i & b_j\end{vmatrix}^2.$$

Problem 2. Let r be a fixed natural number ($r \leqq n-1$) and $\Delta^r a_i = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} a_{i+k}$. Find the best constants $C(n, r)$ and $K(n, r)$ in the inequalities

$$C(n, r) \cdot \min_{i=1, \dots, n-r} |\Delta^r a_i| \leqq D_n(a, a) \leqq K(n, r) \cdot \max_{i=1, \dots, n-r} |\Delta^r a_i|.$$

It is true that $C(n, r) = K(n, r) = D_n(e_r, e_r)$ where $e_r = (1^r, 2^r, \dots, n^r)$?

Problem 3. What we can say about (5) if $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are monotonic in the same sense, and moreover they are convex i. e.,

$$a_{k+2} - 2a_{k+1} + a_k \geqq 0 \text{ and } b_{k+2} - 2b_{k+1} + b_k \geqq 0?$$

REFERENCES

1. D. S. Mitrinonić (in cooperation with P. M. Vasić): *Analytic Inequalities*. Berlin — Heidelberg — New York, 1970.

2. D. S. Mitrinović and G. Kalajdžić: *On an inequality*. These Publications № **668** — № **715** (1980), 3—10.

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O JEDNOJ NEJEDNAKOSTI

A. Lupas

U ovom radu, polazeći od nejednakosti (1), koju je dokazao S. B. Prešić (videti **[1]** p. 341), dokazana je jedna nejednakost koja je analogna sa (1). Osnovni rezultat rada je sadržan u teoremi 2, koja je bazirana na pomoćnom rezultatu teoreme 1. Na kraju rada su navedena i tri problema koja autor smatra nerešenim.