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719. ON SOME FUNCTIONAL EQUATIONS OF PEXIDER'S TYPE*

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0. The general continuous solution of functional equation

$$(1) \qquad f(x_1+\cdots+x_n)=f_1(x_1)+\cdots+f_n(x_n)$$

is (see [1]):

$$f(x)=cx+a_1+\cdots+a_n, \quad f_i(x)=cx+a_i \quad (1\leqq i\leqq n),$$

where $a_i \, (1\leqq i\leqq n)$ are arbitrary real constants. For $n=2$, we have well-known PEXIDER'S equation.

Using this result, we shall give the generalizations for some results from [2].

1. Theorem 1. *The general continuous solution of functional equation*

$$(2) \qquad f(x_1+\cdots+x_n)=I_n(f_1(x_1),\ldots,f_n(x_n);\, p)$$

where

$$I_n(f_1(x_1),\ldots,f_n(x_n);p)=\frac{\sum_{k=1}^{n} kp^{k-1}\sum_{\binom{n}{k}} f_1(x_1)\cdots f_k(x_k)}{1-\sum_{k=2}^{n} p^k(k-1)\sum_{\binom{n}{k}} f_1(x_1)\cdots f_k(x_k)}$$

and $\sum_{\binom{n}{k}} f_1(x_1)\cdots f_k(x_k)$ *denotes the sum of all the combinations of class k of set* $\{f_1(x_1),\ldots,f_n(x_n)\}$*; is*

$$(3) \qquad f(x)=-\frac{1}{p}\,\frac{cx+a_1+\cdots+a_n}{cx+a_1+\cdots+a_n-1}, \quad f_i(x)=-\frac{1}{p}\,\frac{cx+a_i}{cx+a_i-1} \quad (1\leqq i\leqq n),$$

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or

(4) $f(x)=-\frac{1}{p},\ f_k(x)=-\frac{1}{p},\ f_i\left(\not\equiv -\frac{1}{p},\ i\neq k\right)$ *are arbitrary functions.*

The proof is similar to the proof which is given in [2].

Theorem 2. *The general continuous solution of functional equation*

$$f(x_1\cdots x_n)=I_n(f_1(x_1),\ldots,f_n(x_n);p)$$

is

$$f(x)=-\frac{1}{p}\frac{c\log x+a_1+\cdots+a_n}{c\log x+a_1+\cdots+a_n-1},\quad f_i(x)=-\frac{1}{p}\frac{c\log x+a_i}{c\log x+a_i-1}\quad (1\leqq i\leqq n; x>0),$$

or (4).

Theorem 3. *The general continuous solution of functional equation*

$$f(x_1+\cdots+x_n)=I_n(f(x_1),\ldots,f(x_n);p)$$

is $f(x)=-\frac{1}{p}\frac{cx}{cx-1}$; *and the general continuous solution of functional equation*

$$f(x_1\cdots x_n)=I_n(f(x_1),\ldots,f(x_n);p)$$

is $f(x)=-\frac{1}{p}\frac{c\log x}{c\log x-1}$ $(x>0)$.

2. Theorem 4. *The general continuous solution of functional equation*

(5) $$f(I_n(x_1,\ldots,x_n;p))=f_1(x_1)+\cdots+f_n(x_n)$$

is

(6) $$f(x)=c\frac{px}{px+1}+a_1+\cdots+a_n,\quad f_i(x)=c\frac{px}{px+1}+a_i\quad (1\leqq i\leqq n).$$

Proof. Using the substitutions

$$f(x)=g\left(\frac{px}{px+1}\right),\ f_i(x)=g_i\left(\frac{px}{px+1}\right),\quad u_i=\frac{px}{px+1},$$

functional equation (5) gets a from of (1), so, it is obvious that (6) is its general solution.

Theorem 5. *The general continuous solution of functional equation*

$$f(I_n(x_1,\ldots,x_n;p))=f_1(x_1)\cdots f_n(x_n)$$

is

$$f(x)=a_1\cdots a_n\, e^{\frac{cpx}{px+1}},\ f_i(x)=a_i e^{\frac{cpx}{px+1}}\quad (1\leqq i\leqq n).$$

Theorem 6. *The general continuous solution of functional equation*

$$f(I_n(x_1, \dots, x_n; p)) = f(x_1) + \cdots + f(x_n)$$

is $f(x) = c\frac{px}{px+1}$; *and the general continuous solution of equation*

$$f(I_n(x_1, \dots, x_n; p)) = f(x_1) \cdots f(x_n)$$

is $f(x) = e^{\frac{cpx}{px+1}}$.

3. Theorem 7. *The general continuous solution of functional equation*

$$f(I_n(x_1, \dots, x_n; q)) = I_n(f_1(x_1), \dots, f_n(x_n); p) \tag{7}$$

is

$$\begin{aligned} f(x) &= -\frac{1}{p}\frac{cqx+(qx+1)(a_1+\cdots+a_n)}{cqx+(qx+1)(a_1+\cdots+a_n-1)}, \\ f_i(x) &= -\frac{1}{p}\frac{cqx+a_i(qx+1)}{cqx+(qx+1)(a_i-1)} \quad (1 \leqq i \leqq n), \end{aligned} \tag{8}$$

or (4).

Proof. Using the substitutions

$$g_i\left(\frac{qx}{qx+1}\right) = \frac{pf_i(x)}{pf_i(x)+1}, \quad g\left(\frac{qx}{qx+1}\right) = \frac{pf(x)}{pf(x)+1}, \quad u_i = \frac{qx_i}{qx_i+1},$$

analogously to the proof from [2] we can prove the theorem.

Theorem 8. *The general continuous solution of functional equation*

$$f(I_n(x_1, \dots, x_n; q)) = I_n(f(x_1), \dots, f(x_n); p)$$

is $f(x) = -\frac{1}{p}\frac{cqx}{cqx-qx-1}$.

4. Theorem 9. *The general continuous solution of fuctional equation*

$$f(x_1 + \cdots + x_n) = \sum_{k=1}^{n} p^{k-1} \sum_{\binom{n}{k}} f_1(x_1) \cdots f_k(x_k) \tag{9}$$

is

$$f(x) = \frac{1}{p}(e^{cx+a_1+\cdots+a_n} - 1), \quad f_i(x) = \frac{1}{p}(e^{cx+a_i} - 1) \quad (1 \leqq i \leqq n). \tag{10}$$

Proof. Using the substitutions

$$pf(x) + 1 = g(x) \text{ and } pf_i(x) + 1 = g_i(x) \quad (1 \leqq i \leqq n)$$

(9) becomes

$$g(x_1+\cdots+x_n)=g_1(x_1)\cdots g_n(x).$$

This equation has the following general and continuous solution

$$g(x)=e^{cx+a_1+\cdots+a_n}, \quad g_i(x)=e^{cx+a_i} \quad (1\leqq i\leqq n),$$

so, it is obvious that (10) is solution of (9).

Theorem 10. *The general continuos solution of functional equation*

$$f(x_1\cdots x_n)=\sum_{k=1}^{n} p^{k-1}\sum_{\binom{n}{k}} f_1(x_1)\cdots f_k(x_k).$$

is

$$f(x)=\frac{1}{p}(a_1\cdots a_n x^c-1), \quad f_i(x)=\frac{1}{p}(a_i x^c-1) \quad (1\leqq i\leqq n, \quad x>0).$$

Theorem 11. *The general continuous solution of functional equation*

$$f(x_1+\cdots+x_n)=\sum_{k=1}^{n} p^{k-1}\sum_{\binom{n}{k}} f(x_1)\cdots f(x_k)$$

is $f(x)=\frac{1}{p}(e^{cx}-1)$, *and the general continuous solution of functional equation*

$$f(x_1\cdots x_n)=\sum_{k=1}^{n} p^{k-1}\sum_{\binom{n}{k}} f(x_1)\cdots f(x_k)$$

is $f(x)=\frac{1}{p}(x^c-1), \ x>0.$

5. The following two theorems are given in [1]:

Theorem 12. *The general continuous solution of functional equation*

$$f\left(\sum_{k=1}^{n} p^{k-1}\sum_{\binom{n}{k}} x_1\cdots x_k\right)=f_1(x_1)+\cdots+f_n(x_n)$$

is

$$f(x)=c\log(1+px)+a_1+\cdots+a_n, \ f_i(x)=c\log(1+px)+a_i\,(1\leqq i\leqq n)\left(x>-\frac{1}{p}\right).$$

Theorem 13. *The general continuous solution of functional equation*

$$f\left(\sum_{k=1}^{n} p^{k-1}\sum_{\binom{n}{k}} x_1\cdots x_k\right)=f_1(x_1)\cdots f_n(x_n)$$

is

$$f(x)=a_1\cdots a_n(1+px)^c, \ f_i(x)=a_i(1+px)^c \ (1\leqq i\leqq n) \ \left(x>-\frac{1}{p}\right).$$

6. Theorem 14. *The general continuous solution of functional equation*

$$(11)\qquad f\left(\sum_{k=1}^{n} p^{k-1} \sum_{\binom{n}{k}} x_1\cdots x_k\right)=\sum_{k=1}^{n} q^{k-1} \sum_{\binom{n}{k}} f_1(x_1)\cdots f_k(x_k)$$

is

$$(12)\qquad \begin{aligned} f(x)&=\frac{1}{q}(a_1\cdots a_n(px+1)^c-1),\\ f_i(x)&=\frac{1}{q}(a_i(px+1)^c-1)\quad (1\leqq i\leqq n)\qquad \left(x>-\frac{1}{p}\right). \end{aligned}$$

Proof. By substitutions

$$qf(x)+1=g(px+1),\ \ qf_i(x)+1=g_i(px+1),\quad u_i=px_i+1\quad (1\leqq i\leqq n)$$

from (11) we have

$$g(u_1\cdots u_n)=g_1(u_1)\cdots g_n(u_n).$$

This equation has the following general continuous solution:

$$g(u)=a_1\cdots a_n u^c,\quad g_i(u)=a_i u^c\quad (1\leqq i\leqq n),$$

so, it is obvious that (12) is solution of (11).

Theorem 15. *The general continuous solution of functional equation*

$$f\left(\sum_{k=1}^{n} p^{k-1} \sum_{\binom{n}{k}} x_1\cdots x_k\right)=\sum_{k=1}^{n} q^{k-1} \sum_{\binom{n}{k}} f(x_1)\cdots f(x_k)$$

is $f(x)=\frac{1}{q}(px+1)^c-\frac{1}{q}\quad \left(x>-\frac{1}{p}\right)$.

7. Theorem 16. *Functional equation*

$$(13)\qquad f(x_1+\cdots+x_n)=J_n(f_1(x_1),\ldots,f_n(x_n),\varepsilon)$$

where

$$J_n(f_1(x_1),\ldots,f_n(x_n),\varepsilon)=\frac{\sum_{i=0}^{\left[\frac{n-1}{2}\right]} \varepsilon^k \sum_{\binom{n}{2k+1}} f_1(x_1)\cdots f_{2k+1}(x_{2k+1})}{1+\sum_{i=1}^{\left[\frac{n}{2}\right]} \varepsilon^k \sum_{\binom{n}{2k}} f_1(x_1)\cdots f_{2k}(x_{2k})}$$

has a solution:

a) for $\varepsilon=-1$,

(14) $$f(x)=\operatorname{tg}\left(\alpha x+\sum_{i=1}^{n} c_i\right),\quad f_i(x)=\operatorname{tg}(\alpha x+c_i)\quad (1\leqq i\leqq n);$$

b) for $\varepsilon=1$,

(15) $$f(x)=\operatorname{th}\left(\alpha x+\sum_{i=1}^{n} c_i\right),\quad f_i(x)=\operatorname{th}(\alpha x+c_i)\quad (1\leqq i\leqq n)$$

or

(16) $$f(x)=1,\quad f_j(x)=1,\quad f_i(x)\quad (1\leqq i\leqq n,\ i\neq j)\ \textit{arbitrary};$$

or

(17) $f(x)=-1$, $f_j(x)=-1$, $f_i(x)$ $(1\leqq i\leqq n,\ i\neq j)$ *arbitrary.*

Theorem 17. *Functional equation*

(18) $$f(x_1\cdots x_n)=J_n(f_1(x_1),\ldots,f_n(x_n),\varepsilon)$$

has a solution:

a) for $\varepsilon=-1$,

$$f(x)=\operatorname{tg}\left(\alpha\log x+\sum_{i=1}^{n} c_i\right),\quad f_i(x)=\operatorname{tg}(\alpha\log x+c_i)\qquad (1\leqq i\leqq n);$$

b) for $\varepsilon=1$,

$$f(x)=\frac{1-c_1c_2\cdots c_n x^{\alpha}}{1+c_1c_2\cdots c_n x^{\alpha}},\qquad f_i(x)=\frac{1-c_i x^{\alpha}}{1+c_i x^{\alpha}}\qquad (1\leqq i\leqq n)$$

or (16) *or* (17).

Theorem 18. *Functional equation*

(19) $$f(J_n(x_1,\ldots,x_n,\varepsilon))=f_1(x_1)+\cdots+f_n(x_n)$$

has a solution:

a) for $\varepsilon=-1$,

$$f(x)=c\operatorname{arctg}x+\sum_{i=1}^{n} c_i,\quad f_i(x)=c\operatorname{arctg}x+c_i\quad (1\leqq i\leqq n);$$

b) for $\varepsilon=1$,

$$f(x)=c\log\left(\frac{1+x}{1-x}\right)+\sum_{i=1}^{n} a_i,\quad f_i(x)=c\log\left(\frac{1+x}{1-x}\right)+a_i\quad (1\leqq i\leqq n).$$

Theorem 19. *Functional equation*

$$f(J_n(x_1, \ldots, x_n, \varepsilon)) = f_1(x_1) \cdots f_n(x_n) \tag{20}$$

has a solution:

a) for $\varepsilon = -1$,

$$f(x) = e^{c \operatorname{arctg} x + c_1 + \cdots + c_n}, \quad f_i(x) = e^{c \operatorname{arctg} x + c_i} \quad (1 \leqq i \leqq n);$$

b) for $\varepsilon = 1$,

$$f(x) = c_1 \cdots c_n \left(\frac{1+x}{1-x}\right)^c, \quad f_i(x) = c_i \left(\frac{1+x}{1-x}\right)^c \quad (1 \leqq i \leqq n).$$

REMARK 1. The functional equations (13) and (20) are generalizations of some results from [1]. The functional gquation (19) is given in [1].

Theorem 20. *Functional equation*

$$f(J_n(x_1, \ldots, x_n, \varepsilon)) = J_n(f_1(x_1), \ldots, f_n(x_n), \varepsilon) \tag{21}$$

has a solution:

a) for $\varepsilon = -1$,

$$f(x) = \operatorname{tg}(c \operatorname{arctg} x + c_1 + \cdots + c_n), \quad f_i(x) = \operatorname{tg}(c \operatorname{arctg} x + c_i) \quad (1 \leqq i \leqq n);$$

b) for $\varepsilon = 1$,

$$f(x) = \operatorname{th}(c \operatorname{arth} x + c_1 + \cdots + c_n), \quad f_i(x) = \operatorname{th}(c \operatorname{arth} x + c_i) \quad (1 \leqq i \leqq n),$$

or (16) *or* (17).

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O NEKIM FUNKCIONALNIM JEDNAČINAMA PEXIDEROVOG TIPA

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U radu su date dalje generalizacije nekih funkcionalnih jednačina PEXIDERovog tipa (videti [1—3]), tj. posmatrane su jednačine sa proizvoljnim brojem nepoznatih funkcija.