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714. NOTES ON CONVEX FUNCTIONS IV: ON HADAMARD'S INEQUALITIES FOR WEIGHTED ARITHMETIC MEANS*

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In the present paper we give a necessary and sufficient conditions such that the Hadamard's inequalities are valid for an arbitrary arithmetic mean.

1. Let us suppose that p_i $(i=1, \ldots, n)$ are positive constants and let

(1) $$A=\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i},$$

where x_i $(i=1, \ldots, n)$ are given real numbers. Let us suppose further that the function $x \mapsto f(x)$ is convex on the segment $[m, M]$ where we have

(2) $$m=\min_{1\leq i\leq n} \{x_i\}, \quad M=\max_{1\leq i\leq n} \{x_i\}.$$

In the present paper we will investigate for which real values of $y\neq 0$ the following inequalities

(3) $$f\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}\right)\leq\frac{1}{2y}\int_{A-y}^{A+y} f(t)\,\mathrm{d}t\leq\frac{\sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} p_i}$$

are valid for given values of the weights p_i and real numbers x_i.

The proposed problem was considered for $n=2$ in the following papers [1], [2], [3], [4] and [5]. Namely, in papers [1] and [2] we have proved the following theorem.

Theorem 1. *Suppose that $p>0$ and $q>0$ are given constants and suppose that f is two times differentiable convex function on the segment $[a, b]$. Then the inequalities*

(4) $$f\left(\frac{pa+qb}{p+q}\right)\leq\frac{1}{2y}\int_{A-y}^{A+y} f(t)\,\mathrm{d}t\leq\frac{pf(a)+qf(b)}{p+q}$$

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are valid, where

$$A=\frac{pa+qb}{p+q}, \tag{5}$$

if and only if y satisfies the following condition

$$0<|y|\leqq\frac{b-a}{p+q}\min(p,\,q). \tag{6}$$

In paper [3] A. LUPAŞ gave a proof of theorem 1 based on some properties of positive linear operators. The proof of A. LUPAŞ does not require the differentiability of the above function f.

In paper [4] the proof of theorem 1 is given based an approximations of continuous convex functions by polygonal lines.

Paper [5] contains a very short proof of theorem 1 where the only supposition is that the function f is convex on $[a, b]$.

If we have $a_1<a_2<a_3<\cdots<a_{2n}$ and if the function f' is increasing function then the following inequalities

$$f\left(\frac{a_1+a_2+\cdots+a_{2n}}{2n}\right)\leqq\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_{2i}-a_{2i-1}}\int_{a_{2i-1}}^{a_{2i}}f(t)\,dt \tag{7}$$

$$\leqq\frac{f(a_1)+f(a_2)+\cdots+f(a_{2n})}{2n}$$

are valid.

Inequalities (7) are considered in the paper [6].

2. In this part of the present paper we will give a necessary and sufficient conditions for validity of the inequalities (3). First of all we will prove the following lemma.

Lemma 1. *Suppose that p_i $(i=1, \ldots, n)$ are positive constants and that x_i $(i=1, \ldots, n)$ are given real numbers. The inequality*

$$f\left(\frac{\sum_{i=1}^{n}p_i x_i}{\sum_{i=1}^{n}p_i}\right)\leqq\frac{1}{2y}\int_{A-y}^{A+y}f(t)\,dt \tag{8}$$

holds for every convex function f, defined on $[m, M]$ where m and M are given by (2), *if and only if*

$$0<|y|\leqq\min(A-m,\,M-A), \tag{9}$$

where A is defined by (1).

Proof. We have supposed that the function f is convex on $[m, M]$. This means that the same function satisfies the inequality

$$f(A)=f\left(\frac{(A+t)+(A-t)}{2}\right)\leq\frac{f(A+t)+f(A-t)}{2}$$

is valid if and only if $|t|\leq\min(A-m, M-A)$. Therefrom it follows that we have

$$f(A)=\frac{1}{y}\int_0^y f(A)\,\mathrm{d}t\leq\frac{1}{y}\int_0^y\frac{f(A+t)+f(A-t)}{2}\,\mathrm{d}t$$

$$=\frac{1}{2y}\left(\int_A^{A+y} f(t)\,\mathrm{d}t-\int_A^{A-y} f(t)\,\mathrm{d}t\right)=\frac{1}{2y}\int_{A-y}^{A+y} f(t)\,\mathrm{d}t,$$

where we have take $y>0$. This proves the inequality (8) for $y>0$.

But, on the other side the function $y\mapsto F(y)$, defined by

$$F(y)=\frac{1}{2y}\int_{A-y}^{A+y} f(t)\,\mathrm{d}t$$

is even, so that the inequality (8) holds for all y satisfying (9). This completes the proof of lemma 1.

Lemma 2. *Let us suppose that $p_i\,(i=1, \ldots, n)$ are positive constants and let $x_i\,(i=1, \ldots, n)$ be given real numbers. The inequality*

$$\text{(10)}\qquad \frac{1}{2y}\int_{A-y}^{A+y} f(t)\,\mathrm{d}t\leq\frac{\sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} p_i}$$

holds for every function f convex on $[m, M]$, where m and M are given by (2), if and only if

$$\text{(11)}\qquad 0<|y|\leq K=\max_{I\subset I_n}\left\{\frac{\sum_{i\in I} p_i x_i-\sum_{i\in I_n\setminus I} p_i x_i}{\sum_{i=1}^{n} p_i}\min\left\{\sum_{i\in I} p_i,\ \sum_{i\in I_n\setminus I} p_i\right\}\right\}$$

where A is given by (1) and where we have $I_n=\{1, \ldots, n\}$.

Proof. (i) The condition (11) is sufficient.

Let I_0 be one of the subsets of the set I_n for which we have

$$K=\frac{\sum_{i\in I_0} p_i x_i-\sum_{i\in I_n\setminus I_0} p_i x_i}{\sum_{i=1}^{n} p_i}\min\left\{\sum_{i\in I_0} p_i,\ \sum_{i\in I_n\setminus I_0} p_i\right\},$$

where K is define by (11). We will introduce the following denotations

$$a=\frac{\sum_{i\in I_n\setminus I_0} p_i x_i}{\sum_{i\in I_n\setminus I_0} p_i}, \quad b=\frac{\sum_{i\in I_0} p_i x_i}{\sum_{i\in I_0} p_i}, \quad p=\sum_{I\in I_n\setminus I_0} p_i, \quad q=\sum_{I\in I_0} p_i. \tag{12}$$

Since the integral which appears in (10) is even function of the argument y, we will consider only positive values of y which satisfy the condition (11). Those values of y are of the form $y=\frac{b-a}{p+q}z$ where we have $0<z\leqq\min\{p, q\}$ and where a, b, p and q are defined by (12). By using the substitution $x=ta+(1-t)b$ we get

$$\int_{A-y}^{A+y} f(x)\,\mathrm{d}x=(b-a)\int_{\frac{p-z}{p+q}}^{\frac{p+z}{p+q}} f\big(ta+(1-t)b\big)\,\mathrm{d}t$$

$$\leqq(b-a)\int_{\frac{p-z}{p+q}}^{\frac{p+z}{p+q}} (tf(a)+(1-t)f(b))\,\mathrm{d}t$$

$$=2\frac{b-a}{p+q}z\frac{pf(a)+qf(b)}{p+q}=2y\frac{pf(a)+qf(b)}{p+q},$$

which proves that the condition (11) is sufficient.

(ii) The condition (11) is necessary.

Let us suppose that $y>0$ is given in such a way that the inequality (10) is valid, for every function f convex on $[m, M]$. Let us prove that then it must be $y\leqq K$. Moreover, we will consider only those values of y for which $[A-y,\ A+y]\subset[m, M]$. Suppose that $-y\leqq c\leqq A+y$ and that the function f_0 is defined by $f_0(t)=|t-c|$. Since, by our suppositions the inequality (10) holds for an arbitrary convex function f, the same inequality is valid also for the function f_0 just defined. Define the set $J\subset I_n$ in the following way $J=\{i\in I_n \mid x_i\leqq c\}$.

Then immediately follows

$$\frac{1}{2y}\int_{A-y}^{A+y} |t-c|\,\mathrm{d}t=\frac{1}{2y}\Big(\int_{A-y}^{c}(c-t)\,\mathrm{d}t+\int_{c}^{A+y}(t-c)\,\mathrm{d}t\Big)$$

$$=\frac{1}{2y}\left(\left(ct-\frac{t^2}{2}\right)\Big|_{A-y}^{c}+\left(\frac{t^2}{2}-ct\right)\Big|_{c}^{A+y}\right)$$

$$=\frac{1}{2y}\Big(c^2-Ac+cy-\frac{1}{2}(c^2-A^2+2Ay-y^2)+$$

$$+\frac{1}{2}\left(A^2+2\,Ay+y^2-c^2)-cA-cy+c^2\right)$$

$$=\frac{1}{2y}(2c^2-2\,Ac-c^2+A^2+y^2)=\frac{1}{2y}(y^2+A^2-2\,Ac+c^2)$$

$$\leqq\frac{\sum\limits_{i\in J}p_i(c-x_i)+\sum\limits_{i\in I_n\setminus J}p_i(x_i-c)}{\sum\limits_{i=1}^{n}p_i}$$

$$=c\,\frac{\sum\limits_{i\in J}p_i-\sum\limits_{i\in I_n\setminus J}p_i}{\sum\limits_{i=1}^{n}p_i}+\frac{\sum\limits_{i\in I_n\setminus J}p_ix_i-\sum\limits_{i\in J}p_ix_i}{\sum\limits_{i=1}^{n}p_i}=Uc+V,$$

where

$$U=\frac{\sum\limits_{i\in J}p_i-\sum\limits_{i\in I_n\setminus J}p_i}{\sum\limits_{i=1}^{n}p_i},\qquad V=\frac{\sum\limits_{i\in I_n\setminus J}p_ix_i-\sum\limits_{i\in J}p_ix_i}{\sum\limits_{i=1}^{n}p_i}.$$

Finaly we have $\frac{1}{2y}(y^2+A^2-2\,Ac+c^2)\leqq Uc+V$, i.e. we obtain

$$y^2+A^2-2\,Ac+c^2-2\,Uyc-2\,Vy=c^2-2\,c\,(A+Uy)+(y^2+A^2-2\,Vy)\leqq 0. \tag{13}$$

The condition, that the inequality is valid for all $c\in[A-y,\,A+y]$, is equivalent with the condition that (13) is valid for $c=A-y$ and $c=A+y$. The substitution $c=A+y$ and $c=A-y$ in relation (13) leads us to the following relations

$$A^2\mp 2\,Ay+y^2+A^2-2\,yV+y^2-2\,A^2-2\,AUy\pm 2\,Ay\pm 2\,Uy^2$$
$$=2\,y(y\,(1\pm U)-AU-V)\leqq 0$$

which on the basis of the supposition that y is positive gives

$$y\leqq\min\left(\frac{AU+V}{1+U},\ \frac{AU+V}{1-U}\right). \tag{14}$$

Denote $P(I)=\sum\limits_{i\in I}p_i$ and $X(I)=\sum\limits_{i\in I}p_ix_i$ where $I\subset I_n$. Using the definitions of A, U and V from (14) we obtain

$$y\leqq\min\left\{\frac{\frac{X(I_n)}{P(I_n)}\,\frac{P(J)-(P(I_n)-P(J))}{P(I_n)}+\frac{(X(I_n)-X(J))-X(J)}{P(I_n)}}{1\pm\frac{P(J)-(P(I_n)-P(J))}{P(I_n)}}\right\}$$

$$= \min \left\{ \frac{X(I_n)(P(J)-(P(I_n)-P(J)))+P(I_n)((X(I_n)-X(J))-X(J))}{P(I_n)(P(I_n)\pm(P(J)-(P(I_n)-P(J))))} \right\}$$

$$= \min \left\{ \frac{X(I_n)(2P(J)-P(I_n))+P(I_n)(X(I_n)-2X(J))}{2P(J)P(I_n)}, \right.$$

$$\left. \frac{X(I_n)(P(I_n)-2(P(I_n)-P(J)))+P(I_n)(2(X(I_n)-X(J))-X(I_n))}{2P(I_n)(P(I_n)-P(J))} \right\}$$

$$= \min \left\{ \frac{X(I_n)-X(J)}{P(I_n)-P(J)} - \frac{X(I_n)}{P(I_n)},\ \frac{X(I_n)}{P(I_n)} - \frac{X(J)}{P(J)} \right\}$$

$$= \min \left\{ \frac{\sum\limits_{i\in I_n\setminus J} p_i x_i}{\sum\limits_{i\in I_n\setminus J} p_i} - A,\quad A - \frac{\sum\limits_{i\in J} p_i x_i}{\sum\limits_{i\in J} p_i} \right\}$$

$$= \frac{\dfrac{\sum\limits_{i\in I_n\setminus J} p_i x_i}{\sum\limits_{i\in I_n\setminus J} p_i} - \dfrac{\sum\limits_{i\in J} p_i x_i}{\sum\limits_{i\in J} p_i}}{\sum\limits_{i=1} p_i} \min \Big(\sum_{i\in J} p_i,\ \sum_{i\in I_n\setminus J} p_i\Big)$$

$$\leq \max \left\{ \frac{\dfrac{\sum\limits_{i\in I} p_i x_i}{\sum\limits_{i\in I} p_i} - \dfrac{\sum\limits_{i\in I_n\setminus I} p_i x_i}{\sum\limits_{i\in I_n\setminus I} p_i}}{\sum\limits_{i=1}^{n} p_i} \min \Big(\sum_{i\in I} p_i,\ \sum_{i\in I_n\setminus I} p_i\Big) \right\} = K,$$

which proves the lemma.

Using our lemmas 1 and 2 we directly obtain the following theorem:

Theorem 2. *Let us suppose that the function f is convex on $[m, M]$, where m and M are given by* (2). *Then the inequalities* (3) *are valid for every such a function if and only if the condition* (11) *is satisfied.*

The proof of theorem 2 can also be obtained by using the method of positive linear operators on a cone of convex functions.

REMARK. It can be easily shown that there exists at least one $I \subset I_n$ such that the maximal value which appears in (11) is strictly positive.

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NOTE O KONVEKSNIM FUNKCIJAMA IV: O HADAMARDOVIM NEJEDNAKOSTIMA ZA TEŽINSKE ARITMETIČKE SREDINE

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U radu su određeni potrebni i dovoljni uslovi pod kojima važe nejednakosti (3) za proizvoljnu konveksnu funkciju definisanu na $[m, M]$ gde su m i M definisani sa (2). Potrebni i dovoljni uslovi su dati sa (11). Osnovni rezultat rada je sadržan u teoremi 2.