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709. SOME QUADRATURE FORMULAS FOR ANALYTIC FUNCTIONS*

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In [1] the following formula for the numerical quadrature of analytic functions was derived:

$$(1)\qquad \int_{-1}^{1} f(z)\,\mathrm{d}z = Af(0) + B\left(f(k) + f(-k)\right) + C\left(f(ik) + f(-ik)\right) + R,$$

where

$$A = 2\left(1 - \frac{1}{5\,k^4}\right),\qquad B = \frac{1}{6\,k^2} + \frac{1}{10\,k^4},\qquad C = -\frac{1}{6\,k^2} + \frac{1}{10\,k^4}\qquad (k>0)$$

and where the error-term was given by the expansion

$$R = \left(-\frac{2}{3\cdot 6!}k^4 + \frac{2}{7!}\right) f^{(6)}(0) + \left(\frac{2}{9!} - \frac{2}{5\cdot 8!}k^4\right) f^{(8)}(0) + \ldots$$

For $k=1$ BIRKHOFF-YOUNG's (BY) formula was obtained [2]; for $k=\sqrt{0.6}$ the GAUSS-LEGENDRE (GL) formula in three points was obtained since $C=0$ for that case; for $k=\sqrt[4]{3/7}$ the maximum accuracy formula, refered to as MF in [1] (modified BIRKHOFF-YOUNG's formula), was derived.

Formula (1) was derived under the assumption of equal absolute values of arguments of function f, i.e. that the corresponding points fall onto the center and vertices of a square in the complex plane. Notice that (1) can be derived using the method of undetermined coefficients, i.e. under the condition that $R=0$ for as high degree polynomials as possible. A more general problem can be formulated as follows:

Determine parameters A, B, C, x_1, x_2 in the formula

$$(2)\qquad \int_{-1}^{1} f(z)\,\mathrm{d}z = Af(0) + B\left(f(x_1) + f(-x_1)\right) + C\left(f(ix_2) + f(-ix_2)\right) + R,$$

where the error-term R is to be annuled for polynomials of the maximum possible degree.

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For power functions of odd degree: $z, z^3, z^5, \ldots$ the error-term R is annuled. If we substitute functions $1, z^2, z^4, z^6$ and z^8 for f in (2), we obtain the following system:

(3)
$$\frac{A}{2}+B+C=1, \quad B x_1^2 - C x_2^2 = \frac{1}{3}, \quad B x_1^4 + C x_2^4 = \frac{1}{5},$$
$$B x_1^6 - C x_2^6 = \frac{1}{7}, \quad B x_1^8 + C x_2^8 = \frac{1}{9}.$$

REMARK 1. Coefficients of the MF formula are solutions of the first four equations of this system, where $x_1 = x_2$.

System (3) has no real solutions. One of its solutions corresponds to the GL formula in five points, when A, B, C, x_1 are real and x_2 purely imaginary, i.e. when the points are chosen on the real axis.

Solving the first four equations as a system, where it is the most suitable to choose x_2 as a parameter, we get:

$$x_1 = \left(\frac{\frac{1}{5} x_2^2 + \frac{1}{7}}{\frac{1}{3} x_2^2 + \frac{1}{5}}\right)^{1/2}, \qquad B = \frac{\frac{1}{3} x_2^2 + \frac{1}{5}}{x_1^2 (x_1^2 + x_2^2)},$$

$$C = \frac{B x_1^2 - \frac{1}{3}}{x_2^2}, \qquad A = 2(1 - B - C).$$

The error-term can be given in the form of TAYLOR series at $z=0$, where the lead term R_1 is given by

(4)
$$R_1 = \frac{2}{8!}\left(\frac{1}{9} - B x_1^8 - C x_2^8\right) f^{(7)}(0).$$

Parameter x_2 can be determined so that the coefficient multiplying $f^{(7)}(0)$ is minimized. This coefficient is a monotonically increasing function of x_2. When x_2 tends to $+\infty$, then $C \to 0$, $x_1 \to \sqrt{0,6}$, $B \to \frac{5}{9}$, $A \to \frac{8}{9}$ and the GL formula in three points is obtained.

It is convenient to take $x_2 = 0.1$. Then using (3) we get

$$A = 11.58360\,728, \qquad B = 0.39508\,64972,$$
$$C = -\ 5.18689\,0135, \qquad x_1 = 0.84404\,51279,$$
$$R_1 = 4.6338 \cdot 10^{-7} \cdot f^{(7)}(0).$$

This error-term is about 2.8 times smaller than the corresponding term in MF. For $x_2 < 0.1$ the absolute values of coefficients A and C increase. Therefore the obtained formulas are not suitable due to a possible increasing of roundoff errors.

By an analogous procedure we can develop the more general formula which includes BY and MF, as follows:

$$\int_{-1}^{1} f(z)\,dz = Af(0) + B\,(f(x_1) + f(-x_1)) + C\,(f(ix_1) + f(-ix_1)) + D\,(f(x_2) + f(-x_2)) + R. \tag{5}$$

By setting in (5) $f(z)=1, z, z^2, \ldots, z^8$, we obtain the system:

$$\frac{A}{2} + B + C + D = 1, \qquad B\,x_1^2 - C\,x_1^2 + D\,x_2^2 = \frac{1}{3},$$

$$B\,x_1^4 + C\,x_1^4 + D\,x_2^4 = \frac{1}{5}, \qquad B\,x_1^6 - C\,x_1^6 + D\,x_2^6 = \frac{1}{7},$$

$$B\,x_1^8 + C\,x_1^8 + D\,x_2^8 = \frac{1}{9}.$$

For example, if $x_1 = 1$, we get

$$x_2 = 6.83130\,0511 \cdot 10^{-1}, \qquad A = 7.83673\,4695 \cdot 10^{-1},$$

$$B = 8.80952\,3810 \cdot 10^{-2}, \qquad C = -1.73160\,1731 \cdot 10^{-2},$$

$$D = 5.21799\,6289 \cdot 10^{-1}.$$

For these coefficients using (5) we obtain in a simple case $I = \int_{-1}^{1} e^x\,dx \approx 2.35040\,2393$, with the error about $5.77 \cdot 10^{-9}$.

REMARK 2. It is of interest to generalize formulas (2) and (5), by increasing the number of points.

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NEKE KVADRATURNE FORMULE ZA ANALITIČKE FUNKCIJE

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U radu su izvedene dve kvadraturne formule za integraciju analitičkih funkcija. U ovim formulama se pojavljuju argumenti koji ne pripadaju segmentu integracije na realnoj osi, već su čisto imaginarni. Dobijene formule predstavljaju uopštenja do sada poznatih formula. U radu je sugerirana dalja generalizacija.