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708. VERTEX MINIMAL PLANAR CYCLIC GRAPHS*

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ABSTRACT. **Let $c(k)$ $(c^*(k))$ be the least number of vertices of a planar (planar connected) graph whose automorphism group is the cyclic group of order k. If k is odd and if $k = p_1^{a_1} \ldots p_r^{a_r}$, $p_1, \ldots, p_r$ being primes then $c(k) = 3 \sum_{i=1}^{r} p_i^{a_i}$ and $c^*(k) = c(k)$ for $r = 1$ while $c^*(k) = c(k) + 1$ for $r > 1$. It is conjectured that a similar result holds for k even.**

1. Introduction. By a graph we mean a finite undirected loopless graph without multiple edges.

In 1938 R. FRUCHT showed that for any finite froup A there is a graph whose automorphism group is isomorphic to A. This result implied the so called extremum problems: given a finite group A what is the least number of vertices or edges that a graph can have and have automorphism group isomorphic to A?

In 1963 L. R. MERIWETHER (unpublished, see [5] or [7]) completely answered the vertex extremum problem for cyclic groups of all finite orders.

In this paper we shall be mainly concerned with a modified form of the above vertex extremum problem imposing an additional constraint upon the planarity of the graph in question.

Let us call a graph k-cyclic if its automorphism group is isomorphic to the cyclic group of order k. Let $c(k)$ $(c^*(k))$ denote the number of vertices of a vertex minimal planar (planar connected) k-cyclic graph, that is $c(k)$ $(c^*(k))$ is the least number of vertices for which a planar (planar connected) k-cyclic graph exists.

The main aim of this paper is to find the exact values of $c^*(k)$ and $c(k)$ or at least upper bounds.

2. Star polygons and chains. By Z_n we shall denote the set of integers modulo n. G will always denote a graph and G^c, $V(G)$, $E(G)$, $\operatorname{Aut} G$ will denote the complement, the vertex set, the edge set, the automorphism group of G, respectively. The subgraph of G induced by a vertex subset X will be denoted by X^*. Let X and Y be any two desjoint subsets of $V(G)$, by $L(X, Y)$ we shall denote the subgraph of G with the vertex set $X \cup Y$ and the edge set $E((X \cup Y)^*) \setminus E(X^*) \cup E(Y^*)$.

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Let A be any subgroup of $\operatorname{Aut} G$. A nonempty subset $W \subseteq V(G)$ is an A-orbit if $f(W)=W$ for each $f \in A$ and if for every nonempty proper subset U of W there exists at least one $f \in A$ such that $f(U) \neq U$. Let C be a union of some A-orbits. Then each $f \in A$ induces an automorphism of C^* which we shall denote by f_C. By $\langle f_1, \dots, f_r \rangle$ we shall denote the subgroup of $\operatorname{Aut} G$ generated by the automorphisms $f_1, \dots, f_r$.

An automorphism g of G is called cyclic if $g \in \langle f \rangle$ for some automorphism f of maximum order of G and noncyclic if it is not cyclic.

Definition 2.1. *A graph G is an n-star polygon if $|V(G)|=n$ and if* Aut G *contains a cyclic permutation of $V(G)$.*

The automorphism group of an n-star polygon, $n \geqq 3$ contains the dihedral group D_n. For a given star polygon G we define $D(G)$ to be the corresponding dihedral group if $|V(G)| \geqq 3$ and the whole group if $|V(G)|<3$.

Definition 2.2. *A bipartite graph with a bipartition (U, W) is an n, m-chain if $|U|=n \geqq m=|W|$ and if it has an automorphism f such that U and W are $\langle f \rangle$-orbits.*

Clearly, $n \cdot s = m \cdot t$, where s and t are degrees of vertices in U and W respectively. By $B(n, m, s)$, $n \geqq m \geqq s \geqq 0$ we shall denote an n, m-chain whose n vertices have degree s and m vertices degree $n \cdot s/m$ (Note that $s>0$ implies $s \geqq m/(n, m)$ where (n, m) is the g. c. d. of n and m).

We call an n, m-chain divisible if m divides n. We shall use FRUCHT's notation [3] for star polygons[1] and divisible chains.

The next result is immediate:

Proposition 2.3. *Let f be an automorphism of a graph G. Then there exists a decomposition of G into star polygons and chains such that:*

(i) *every star polygon is induced by some $\langle f \rangle$-orbit,*

(ii) *every chain is induced by some pair of $\langle f \rangle$-orbits.*

3. Cyclic graphs. Since K_2 is 2-cyclic and therefore $c(2)=c^*(2)=2$, interesting results only arise for k-cyclic graphs, $k \geqq 3$. Let G be any such graph. By an orbit we shall mean an Aut G-orbit, by f we denote a generator of Aut G.

For any two orbits X, Y, say $|X| \geqq |Y|$, the number $s(X, Y)=s(Y, X)$ equals s iff $L(X, Y)=B(|X|, |Y|, s)$. Two orbits X, Y are a d j a c e n t (or n e i g h b o u r s) if $s(X, Y)>0$. A l o n g orbit is an orbit of cardinality >1. By $\mathscr{L}(G)$ we shall denote the set of all long orbits. By $N(x)$ $(N(X))$ we shall denote the set of neighbours (long neighbours) of a vertex x (orbit X). Furthermore for a given nonempty set of orbits $\mathscr{C}$ let $N(\mathscr{C}) = \bigcup_{Y = \mathscr{C}} N(Y) \setminus \mathscr{C}$.

[1] For techinical reasons, star polygons, when mentioned in the text, will be given by a square instead by a circle.

Definition 3.1. *An orbit is* maximal *if its cardinality is greater than* 2 *and greater than the cardinality of any of its neighbours. An orbit is* nearly maximal *if its cardinality is greater than* 2, *no neighbour has greater cardinality and exactly one neighbour has the same cardinality.*

Definition 3.2. *An unordered pair of orbits* (X, X') *is said to be* divisible *if:*

(i) $Y \in N(\{X, X'\}) \Rightarrow L(X, Y)$ *and* $L(X', Y)$ *are divisible,*

(ii) $Y, Z \in N(\{X, X'\}) \Rightarrow L(Y, Z)$ *is divisible.*

Every orbit of maximum cardinality in $N(\{X, X'\})$ *is called a leader of* (X, X').

Definition 3.3. *An orbit* X *is* divisible *if the pair* (X, X) *is divisible.* Y *is a* leader *of* X *if it is a leader of* (X, X).

Lemma 3.4. *Let* X, Y *be any two orbits such that either* $s(X, Y) \leq 2$ *or* $s(X, Y) \geq \min\{|X|, |Y|\} - 2$. *For every automorphism* $g' \in D(X^*)$ *we can find an automorphism* $g'' \in D(Y^*)$ *such that there is an automorphism* g *of* $(X \cup Y)^*$ *with* $g_X = g'$, $g_Y = g''$.

Proof. We can put $X = \{x, f(x), \dots, f^{m-1}(x)\}$, $Y = \{y, f(y), \dots, f^{n-1}(y)\}$, $s = s(X, Y)$ where $m = |X|$, $n = |Y|$, and y is chosen to be adjacent to x when $1 \leq s \leq 2$, nonadjacent to x when $1 \leq \min\{n, m\} - s \leq 2$ and an arbitrary vertex otherwise. If g' is cyclic then there exists an integer r such that $g' = f_X^r$. In that case put $g'' = f_Y^r$. If g' is noncyclic then there exists an integer r such that g' maps according to the formula: $g'(f^j(x)) = f^{r-j}(x)$ $(j = 0, 1, \dots, m-1)$. If $s \leq 1$ or $\min\{m, n\} - s \leq 1$ then let $g''(f^j(y)) = f^{r-j}(y)$ $(j = 0, 1, \dots, n-1)$. If $s = 2$ $(\min\{m, n\} - s = 2)$ then let $y' = f^q(y)$ be the vertex having the minimum positive exponent q in $Y \cap N(x)$ $(Y \setminus N(x))$. Then let $g''(f^j(y)) = f^{q+r-j}(y)$, $(j = 0, 1, \dots, n-1)$.

Definition 3.5. *A set of orbits* $\mathscr{C}$ *is* stout *if* $\mathscr{C} \subseteq \mathscr{L}(G)$, $N(\mathscr{C}) = \emptyset$ *and if for every proper nonempty subset* $\mathscr{C}' \subset \mathscr{C}$ *it follows that* $N(\mathscr{C}') \neq \emptyset$.

It is not hard to see that $\mathscr{L}(G)$ is partitioned into stout disjoint subsets.

Definition 3.6. *A set of orbits* $\mathscr{C}$ *is* cyclic *if* $\mathscr{C} \subseteq \mathscr{L}(G)$ *and if either*

(i) $\mathscr{C} = \{X, Y\}$ *and* $3 \leq s(X, Y) \leq \min\{|X|, |Y|\} - 3$, *or*

(ii) $\mathscr{C} = \{X_1, \dots, X_r\}$, $r \geq 3$ *and* $s(X_1, X_2), \dots, s(X_r, X_1) \geq 1$.

Proposition 3.7. *Every stout set which contains at least one orbit of cardinality* >2 *contains a cyclic subset.*

Proof. Let $\mathscr{C}$ be a stout set which contains an orbit of cardinality >2, say X and let C denote the union of all orbits in $\mathscr{C}$. Assume that $\mathscr{C}$ does not contain a cyclic subset. Take a noncyclic automorphism $g \in D(X^*)$. It generates a noncyclic automorphism g' of C^* such that $g'_X = g$. Namely, since $\mathscr{C}$ contains no cyclic subset we can apply 3.4. step by step starting with the orbit

X and the automorphism g and constructing g' first on all long neighbours of X, then on all orbits in $N(N(X))$ etc. No orbit is met twice and g' is uniquely determined. Therefore C^* is not cyclic and neither can be G since the permutation h given by:

(i) $h_C = g'$,

(ii) $h_{V(G)\setminus C} = id$

is a noncyclic automorphism of G. This contradiction proves 3.7.

Corollary 3.8. *$\mathscr{L}(G)$ contains at least one cyclic subset.*

Proof. Since G is k-cyclic for some $k \geqq 3$ it follows that $\mathscr{L}(G)$ contains at least one orbit of cardinality >2. The rest follows from 3.7.

Proposition 3.9. *Let X be a maximal divisible orbit of cardinality n. Then X^* contains a subgraph isomorphic to the star polygon $\overline{|n(r)|}$, $r \not\equiv 0 \pmod m$ where m is the cardinality of a leader of X.*

Proof. Since X is maximal and therefore its cardinality >2 it follows by 3.7. that $N(X) \neq \emptyset$ and hence X has a leader, say Y of cardinality m. X decomposes into $q = n/m$ $\langle f^m \rangle$-orbits: $X_0, \ldots, X_{q-1}$ (of cardinality m each). Since X is divisible and Y its leader and also an $\langle f^m \rangle$-orbit it follows that for any two vertices u, v belonging to the same $\langle f^m \rangle$-orbit X_i the sets $N(u)\setminus X$ and $N(v)\setminus X$ are equal. If X^* doesn't contain a subgraph isomorphic to $\overline{|n(r)|}$, $r \not\equiv 0 \pmod m$ then the chain $L(X_i, X_j)$ (for any two distinct i, j) is totally disconnected and therefore the permutation g given by:

(i) $g_{X_0} = g_0$, an arbitrary automorphism of order 2 of X_0^*,

(ii) $g_{V(G)\setminus X_0} = id$

is a noncyclic automorphism of G, a contradiction.

Proposition 3.10. *Let (X, X') be a divisible pair of nearly maximal adjacent orbits such that $s(X, X') < 3$ and let $n = |X| = |X'|$. Then $(X \cup X')^*$ contains a subgraph homeomorphic to the star polygon $\overline{|n(r)|}$, $r \not\equiv 0 \pmod m$ where m is the cardinality of a leader of (X, X').*

Proof. It follows by 3.7. that $N(\{X, X'\}) \neq \varnothing$ and therefore (X, X') has a leader, say Y of cardinality m. X and X' decompose into $q = n/m$ $\langle f^m \rangle$-orbits (each): $X_0, \ldots, X_{q-1}$; $X'_0, \ldots, X'_{q-1}$, respectively. The subscripts are chosen in such a way that $L(X_i, X'_i)$ $(i = 0, \ldots, q-1)$ contains a subgraph isomorphic to mK_2.

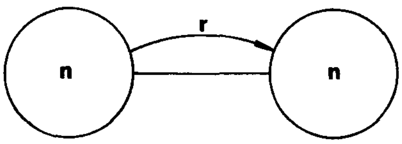
Case 1: $s(X, X')=1$. If neither X^* nor X'^* contains a subgraph isomorphic to $\overline{|n(r)|}$ such that $r \not\equiv 0 \pmod m$ then for any two distinct i, j the chains $L(X_i, X_j)$ and $L(X_i', X_j')$ are totally disconnected. The permutation g given by:

(i) $g_{X_0 \cup X'_0} = g_0$, an arbitrary automorphism of order 2 of $(X_0 \cup X_0')^*$,

(ii) $g_{V(G) \setminus (X_0 \cup X'_0)} = id$

is a noncyclic automorphism of G, a contradiction.

Case 2: $s(X, X')=2$. There is an integer r such that $L(X, X')$ equals



and is therefore homeomorphic to $\overline{|n(r)|}$. If $r \not\equiv 0 \pmod m$ then 3.10. is true. If $r \equiv 0 \pmod m$ then either X^* or X'^* contains a subgraph isomorphic to $\overline{|n(t)|}$, $t \not\equiv 0 \pmod m$ for otherwise we could find a noncyclic automorphism of G as in Case 1. This completes the proof of 3.10.

4. Planar cyclic graphs.

Definition 4.1. *The* c o r n e r s *of a graph homeomorphic to* $K_{3,3}$ *are the six vertices of degree* 3.

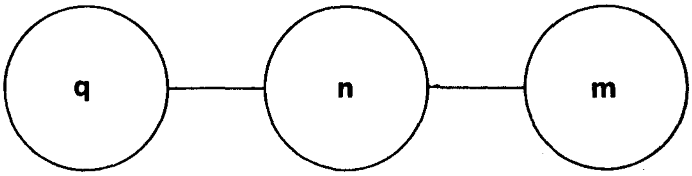
From EULER's formula we derive this necessary condition for planar graphs with e edges, w vertices and each face surrounded with at least a edges:

(1) $$a(w-2) \geqq (a-2)e.$$

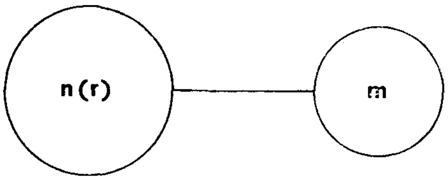
Lemma 4.2. *Every chain* $B(n, n, 3)$, $n \geqq 3$ *odd, is nonplanar.*

Proof. Assume that there exists a planar chain $L=B(n, n, 3)$ for some odd $n \geqq 3$. Let (X, Y) be its bipartition. L contains only even circuits. If it doesn't contain a 4-circuit then (1) implies $12(n-1) \geqq 12n$ which is impossible. Thus L must contain a 4-circuit, say with vertex set $\{x, y, x_1, y_1\}$ where $x, x_1 \in X$, $y, y_1 \in Y$. Let the other neighbours of x and x_1 be y_2 and y_3 respectively. There is an automorphism h of L which takes x into x_1 and fixes X and Y set-wise and such that every $\langle h \rangle$-orbit has cardinality $\geqq 3$. If $y_2 = y_3$ there is a vertex x_2 differenet from x and x_1 such that its neighbours are y, y_1 and y_2. Hence $\{x, x_1, x_2, y, y_1, y_2\}^* = K_{3,3}$ and L nonplanar, a contradiction. Therefore $y_2 \neq y_3$. Since $h: \{y, y_1, y_2\} \to \{y, y_1, y_3\}$ and since every $\langle h \rangle$-orbit has cardinality $\geqq 3$ one of the vertices y and y_1 must be mapped into y_3, e.g. $h(y_1)=y_3$. This implies $h(y)=y_1$, $h(y_2)=y$. Therefore $N(x)=\{h^{-1}(y), y, h(y)\}$ and the neighbours' set of any other vertex of L can be written in an analogous way. It follows that L contains the paths $(h(x), h^2(y), \ldots, h^{-2}(x), h^{-1}(y))$, $(h(y), h^2(x), \ldots, h^{-2}(y), h^{-1}(x))$ and therefore the vertices $x, h(x), h^{-1}(x), y, h(y), h^{-1}(y)$ are the corners of a graph homeomorphic to $K_{3,3}$ and L nonplanar, a contradiction which proves 4.2.

Let $n>m>q\geqq 3$ be odd numbers such that m and q divide n and q does not divide m. By $G(n, m, q)$ we shall denote the graph



Let $n>m\geqq 3$ be odd numbers such that m divides n and let $r\not\equiv 0 \pmod{m}$. By $H(n, m, r)$ we shall denote the graph



Lemma 4.3. *$G(n, m, q)$ is nonplanar.*

Proof. Let $V(G(n, m, q))=\{x_i : i\in Z_n\}\cup\{w_i : i\in Z_m\}\cup\{w_i : i\in Z_q\}$ such that $E(G(n, m, q))=\{(x_i, y_j) : j\equiv i \pmod{m}\}\cup\{(x_i, w_j) : j\equiv i \pmod{q}\}$. Since $G(n, m, q)$ contains the paths: (y_0, x_0, w_0), (y_0, x_m, w_m), (y_0, x_{2m}, w_{2m}), (y_q, x_q, w_0), (y_q, x_{m+q}, w_m), $(y_q, x_{2m+q} w_{2m})$, (y_{2q}, x_{2q}, w_0), (y_{2q}, x_{m+2q}, w_m), $(y_{2q}, x_{2m+2q}, w_{2m})$, the vertices $y_0, y_q, y_{2q}, w_0, w_m, w_{2m}$ are the corners of a graph homeomorphic to $K_{3,3}$ and thus $G(n, m, q)$ is nonplanar.

Lemma 4.4. *$H(n, m, r)$ is nonplanar.*

Proof. Let $V(H(n, m, r)) = \{x_i : i \in Z_n\} \cup \{y_i : i \in Z_m\}$ such that $E(H(n, m, r)) = \{(x_i, x_{i+r}) : i \in Z_n\} \cup \{(x_i, y_j) : j\equiv i \pmod{m}\}$. Since $H(n, m, r)$ contains the paths: (y_0, x_0), (y_0, x_m), (y_0, x_{2m}), (y_r, x_r, x_0), (y_r, x_{m+r}, x_m), (y_r, x_{2m+r}, x_{2m}), (y_{-r}, x_{-r}, x_0), (y_{-r}, x_{m-r}, x_m), $(y_{-r}, x_{2m-r}, x_{2m})$, the vertices $x_0, x_m, x_{2m}, y_0, y_r, y_{-r}$ are the corners of a graph homeomorphic to $K_{3,3}$ and therefore $H(n, m, r)$ is nonplanar.

Let G be a planar k-cyclic graph for some odd $k\geqq 3$. By an orbit we shall mean an Aut G-orbit.

Proposition 4.5. *Let X, Y be any two adjacent orbits. Then $L(X, Y)$ is divisible and $s(X, Y)\leqq 2$.*

Proof. Let $|X|=n$, $|Y|=m$. If $m=n$ then $L(X, Y)$ is divisible. If $m\neq n$, say $m<n$, and m doesn't divide n then (since m and n are odd) $e=n/(n, m)>d= =m/(n, m)\geqq 3$. Therefore $L(X, Y)$ contains a nonplanar subgraph $K_{e,d}$, a contradiction. Thus m divides n and $L(X, Y)$ is divisible. It follows that $L(X, Y)$ contains a chain $B(m, m, s(X, Y))$ and therefore by 4.2, $s(X, Y)\leqq 2$.

Proposition 4.6. *No orbit is maximal.*

Proof. Suppose that there exists a maximal orbit X. We claim that X is divisible. If it isn't then there exist $Z, W\in N(X)$, say $|Z|>|W|$, such that $|W|$ doesn't divide $|Z|$. By 4.5. it follows that $|Z|$ and $|W|$ divide $|X|$ and therefore $(X\cup Z\cup W)^*$ contains the subgraph $G(|X|, |Z|, |W|)$ which is by 4.3.

nonplanar, a contradiction. Thus X is divisible. Since $N(X) \neq \varnothing$ it follows that X has a leader, say Y of cardinality $\geqq 3$. By 3.9. there exists an integer r such that $(X \cup Y)^*$ contains the subgraph $H(|X|, |Y|, r)$ which is by 4.4. nonplanar, a contradiction.

Proposition 4.7. *Two nearly maximal orbits are not adjacent.*

Proof. Suppose that there exist two nearly maximal adjacent orbits X and X'. Clearly $|X| = |X'|$ and by 4.5. it follows that $1 \leqq s(X, Y) \leqq 2$. We claim that (X, X') is divisible. If it isn't then there exist two orbits $Z, W \in N(X, X')$, say $|Z| > |W|$, such that $|W|$ doesn't divide $|Z|$. Therefore $(X \cup X' \cup Z \cup W)^*$ contains a subgraph homeomorphic to $G(|X|, |Z|, |W|)$ which is by 4.3. nonplanar. Thus (X, X') is divisible. Let Y be a leader of (X, X'). By 3.10. it follows that there is an integer r such that $(X \cup X' \cup Y)^*$ contains a subgraph homeomorphic to $H(|X|, |Y|, r)$ which is by 4.4. nonplanar, a contradiction.

Proposition 4.8. *There exist integers $k_1, \dots, k_t$, $t \geqq 1$, such that:*

(2) *the l. c. m. $[k_1, \dots, k_t]$ equals k and for each $i \in \{1, \dots, t\}$ there are at least three orbits X_i, X_i' X_i'' of cardinality k_i.*

Proof. Let k_1 be the maximum of all orbits' cardinalities. It follows by 4.6. and 4.7. that there must exist at least three orbits of cardinality k_1, say X_1, X_1', X_1''. If $k_1 = k$ then 4.8. is true. If not then (since the *l. c. m.* of all orbits' cardinalities must be k) there exists at least one orbit of cardinality not dividing k_1 and therefore not in $N(\{X_1, X_1', X''\})$. Let k_2 be the maximum of cardinalities of all such orbits. Then again by 4.6. and 4.7. there exist at least three orbits of cardinality k_2, say X_2, X_2', X_2''. If $[k_1, k_2] = k$ then 4.8. is proved. If not we go on with the same argument as long as for some t the *l. c. m.* of $k_1, k_2, \dots, k_t$ equals k.

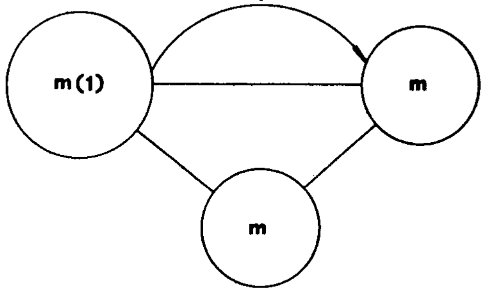
5. Numbers $c(k)$ and $c^*(k)$.

Theorem 5.1. *Let $k \geqq 3$ be an odd number and let $k = p_1^{a_1} \dots p_r^{a_r}$ be its prime powers factorisation. Then:*

(i) $c(k) = 3 \sum_{i=1}^{r} p_i^{a_i}$,

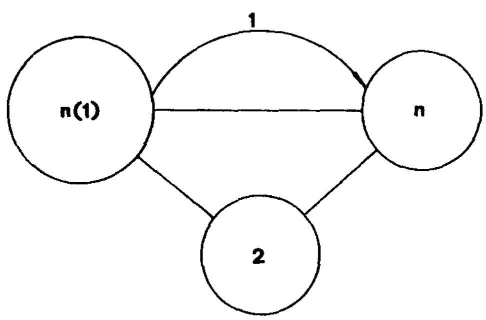
(ii) $c^*(k) = c(k)$ *if* $r = 1$ *and* $c^*(k) = c(k) + 1$ *if* $r > 1$.

Proof. Let $G(m)$, m any integer $\geqq 3$, denote the graph



Clearly $G(m)$ is planar and m-cyclic. Therefore the union $U(k)=G(p_1^{a_1})\cup \cdots \cup G(p_r^{a_r})$ is a planar k-cyclic graph. Let G be any planar k-cyclic graph and let $k_1, \ldots, k_t$ be the integers satisfying (2). It is not difficult to convince oneself that $\sum_{i=1}^{t} k_i \geqq \sum_{i=1}^{r} p_i^{a_i}$ and that the equality holds iff k_i's are the prime powers $p_i^{a_i}$'s. Hence $|V(G)|\geqq V(U(k))|$ and $U(k)$ is vertex minimal. This proves (i). If $r=1$ then it is also connected. If $r>1$ then we need another vertex to „make" it connected. We join this vertex by an edge to all vertices that have degree 5 in $U(k)$. The so defined graph is then vertex minimal planar connected k-cyclic and this completes the proof of 5.1.

Let $H(n)$, n any even number, denote the graph



when $n\geqq 4$ and K_2 when $n=2$. Clearly $H(n)$ is planar n-cyclic. Since R. L. MERIWETHER proved that a 4-cyclic graph has at least 10 vertices it follows that $H(2)$ ($H(4)$) is vertex minimal planar 2-cyclic (4-cyclic). Our conjecture is that the same is true for any $H(2^a)$, $a\geqq 1$. (This is very likely to be true since it was pointed out in a letter to the author by L. BABAI that the graph in Fig. 1. is nonplanar which was the toghest obstacle in one's attempt to prove the above stated conjecture). Since all these graphs are connected our conjecture extends to say that $H(2^a)$, $a\geqq 1$, is also vertex minimal planar connected 2^a-cyclic.

When k is an even number but not a power of 2, say $k=2^a\cdot m$, $a\geqq 1$, m odd>1, then we conjecture that $H(2^a)\cup U(m)$ is vertex minimal planar k-cyclic. When $a=1$ we get the vertex minimal connected graph by joining all vertices of degree 5 in $U(m)$ and the two vertices of $H(2)$ to a new vertex. When $a>1$ this construction doesn't give us a planar graph and we conjecture that we have to use the graph $G(2^a)$ instead of $H(2^a)$.

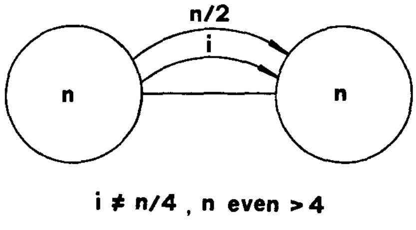


Fig. 1.

6. A conjecture about edge minimality. In [1] R. FRUCHT and I. Z. BOUWER conjectured that for every prime p the graphs in Fig. 2. are edge minimal p^a-cyclic. It can be seen that this is true and that this is a quite straightforward consequence of 3.7, 3.9, 3.10. However, as was mentioned by L. V. QUINTAS (personal correspondence to the author), the edge extremum problem for cyclic groups has finally been solved, a solution is to appear in a joint paper by L. V. QUINTAS and DON MC CARTHY.

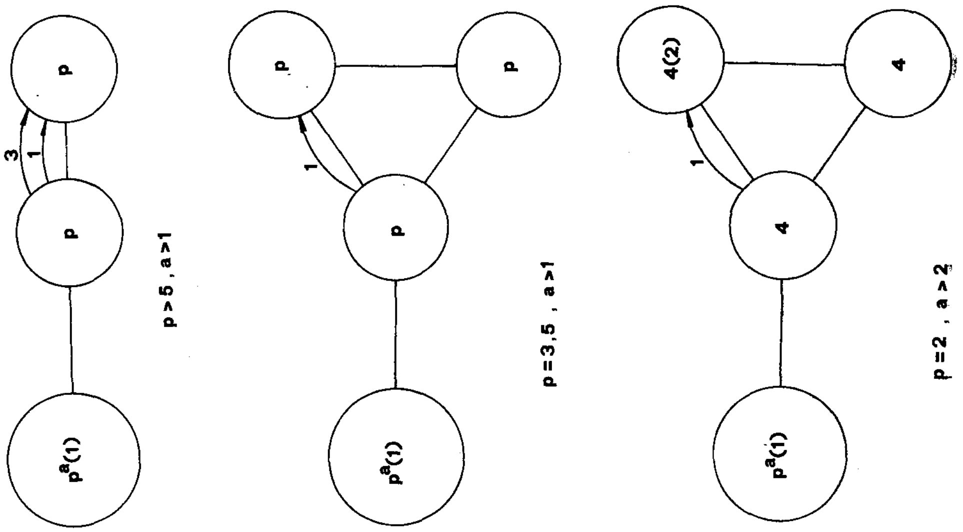


Fig. 2.

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ČVORNO MINIMALNI PLANARNI CIKLIČKI GRAFOVI

D. Marušič

Neka je $c(k)$ ($c^*(k)$) najmanji broj takav da postoji planaran (planaran povezan) graf sa $c(k)$ ($c^*(k)$) čvorova čija je grupa automorfizama ciklička grupa reda k. Ako je k neparno i $k = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, gde su $p_1, \ldots, p_r$ prosti brojevi, tada je $c(k) = 3 \sum_{i=1}^{r} p_i^{\alpha_i}$ i $c^*(k) = c(k)$ za $r=1$, dok $c^*(k) = c(k)+1$ za $r>1$.