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684. A NOTE ON SECOND ORDER INTEGRODIFFERENTIAL INEQUALITIES OF THE GRONWALL-BELLMAN TYPE*

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The present note deals with some new second order integrodifferential inequalities of the Gronwall-Bellman type which can be used in the analysis of a class of integrodifferential equations as handy tools.

1. Introduction. Integral inequalities associated with the names of GRONWALL [3], BELLMAN [1], and BIHARI [2] have played a fundamental role in the study of differential and integral equations. The integral inequalities of the GRONWALL-BELLMAN type recently established by this author in [4]—[10] have been applied with considerable success to the study of many problems in the theory of differential and integral equations of the more general type. In this note we wish to establish some new second order integrodifferential inequalities which can be used in investigating the behavior of solutions of a class of integrodifferential equations.

2. Main Results. In this section we establish our main results on integrodifferential inequalities of the GRONWALL-BELLMAN type which can be used in the study of many nonlinear problems in the theory of integrodifferential equations.

A useful integrodifferential inequality of the GRONWALL-BELLMAN type is embodied in the following theorem.

Theorem 1. *Let $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$, and $a(t)$ be real-valued nonnegative continuous functions defined on $I=[0, \infty)$, for which the inequality*

$$\ddot{x}(t) \leqq x(0)+\dot{x}(0)+\int_0^t a(s)\left(x(s)+\dot{x}(s)+\ddot{x}(s)\right) \mathrm{d}s \tag{1}$$

holds for all $t \in I$. Then

$$\begin{aligned}\ddot{x}(t) \leqq x(0)+\dot{x}(0)+\int_0^t a(s)\Big[2[x(0)+\dot{x}(0)] \exp\Big(\int_0^s a(\tau)\,\mathrm{d}\tau\Big) \\ +\int_0^s [\dot{x}(0)+2[x(0)+\dot{x}(0)]] \exp\Big(\int_0^\tau [2+a(n)]\,\mathrm{d}n\Big) \exp\Big(\int_\tau^s a(n)\,\mathrm{d}n\Big) \mathrm{d}\tau\Big] \mathrm{d}s,\end{aligned} \tag{2}$$

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for all $t \in I$.

Proof. Define

(3) $$m(t)=x(0)+\dot{x}(0)+\int_0^t a(s)\,(x(s)+\dot{x}(s)+\ddot{x}(s))\,ds, \quad m(0)=x(0)+\dot{x}(0),$$

then (1) can be restated as

(4) $$\ddot{x}(t)\leq m(t).$$

Differentiating (3) and using (4) we have

(5) $$\dot{m}(t)\leq a(t)\,(x(t)+\dot{x}(t)+m(t)).$$

Integrating (4) from 0 to t we have

(6) $$\dot{x}(t)\leq \dot{x}(0)+\int_0^t m(s)\,ds,$$

further integrating (6) from 0 to t we see that the inequality

(7) $$x(t)\leq x(0)+\int_0^t \Big[\dot{x}(0)+\int_0^s m(\tau)\,d\tau\Big]\,ds,$$

is satisfied for all $t\in I$. Using (6) and (7) in (5) we have

(8) $$\dot{m}(t)\leq a(t)\Big[x(0)+\dot{x}(0)+m(t)+\int_0^t m(s)\,ds+\int_0^t \Big[\dot{x}(0)+\int_0^s m(\tau)\,d\tau\Big]ds\Big].$$

Define

(9) $$u(t)=x(0)+\dot{x}(0)+m(t)+\int_0^t m(s)\,ds+\int_0^t\Big[\dot{x}(0)+\int_0^s m(\tau)\,d\tau\Big]ds,$$
$$u(0)=x(0)+\dot{x}(0)+m(0),$$

then differentiating (9) and using the fact that $\dot{m}(t)\leq a(t)\,u(t)$ and $m(t)\leq u(t)$ from (8) and (9) we have

(10) $$\dot{u}(t)\leq a(t)\,u(t)+\dot{x}(0)+u(t)+\int_0^t u(\tau)\,d\tau.$$

Put

(11) $$V(t)=\dot{x}(0)+u(t)+\int_0^t u(\tau)\,d\tau, \quad V(0)=\dot{x}(0)+u(0)=\dot{x}(0)+2\,[x(0)+\dot{x}(0)],$$

then differentiating (11) and using the facts that $\dot{u}(t) \leqq a(t)u(t)+V(t)$ and $u(t) \leqq V(t)$ from (10) and (11) we see that the inequality

$$\dot{V}(t) \leqq (2+a(t))\, V(t),$$

is satisfied, which implies the estimation for $V(t)$ such that

$$V(t) \leqq [\dot{x}(0)+2\,[x(0)+\dot{x}(0)]] \exp\Big(\int_0^t [2+a(s)]\, ds\Big).$$

Substituting this value of $V(t)$ in (10) we have

$$\dot{u}(t) \leqq a(t)u(t)+[\dot{x}(0)+2\,[x(0)+\dot{x}(0)]] \exp\Big(\int_0^t [2+a(s)]\, ds\Big)$$

which implies the estimation for $u(t)$ such that

$$u(t) \leqq 2\,[x(0)+\dot{x}(0)] \exp\Big(\int_0^t a(s)\, ds\Big)$$
$$+\int_0^t [\dot{x}(0)+2\,[x(0)+\dot{x}(0)]] \exp\Big(\int_0^s [2+a(s)]\, ds\Big) \exp\Big(\int_s^t a(\tau)\, d\tau\Big)\, ds.$$

Now, substituting the value of $u(t)$ in (8) we see that the inequality

$$\dot{m}(t) \leqq a(t)\Big[2\,[x(0)+\dot{x}(0)] \exp\Big(\int_0^t a(s)\, ds\Big)$$
$$+\int_0^t \Big[\dot{x}(0)+2\,[x(0)+\dot{x}(0)]] \exp\Big(\int_0^s [2+a(\tau)]\, d\tau\Big) \exp\Big(\int_s^t a(\tau)\, d\tau\Big)\, ds\Big]$$

is satisfied, which implies the estimation for $m(t)$ such that

$$m(t) \leqq x(0)+\dot{x}(0)+\int_0^t a(s)\Big[2\,[x(0)+\dot{x}(0)] \exp\Big(\int_0^s a(\tau)\, d\tau\Big)$$
$$+\int_0^s [\dot{x}(0)+2\,[x(0)+\dot{x}(0)]] \exp\Big(\int_0^\tau [2+a(n)]\, dn\Big) \exp\Big(\int_\tau^s a(n)\, dn\Big)\, d\tau\Big]\, ds$$

Now, substituting this value of $m(t)$ in (1) we obtain the desired bound in (2).

Another interesting and slightly different version of the integrodifferential inequality established in Theorem 1 may be stated as follows.

Theorem 2. *Let $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$, and $a(t)$ be real-valued nonnegative continuous functions defined on I, for which the inequality*

$$\ddot{x}(t)+x(t)\leqq x(0)+\dot{x}(0)+\int_0^t a(s)\big(x(s)+\dot{x}(s)+\ddot{x}(s)\big)\,\mathrm{d}s, \tag{12}$$

holds for all $t\in I$. Then, for all $t\in I$,

$$\ddot{x}(t)+x(t)\leqq x(0)+\dot{x}(0)+\int_0^t a(s)\,[x(0)+2\dot{x}(0)]\exp\Big(\int_0^s [1+a(\tau)]\,\mathrm{d}\tau\Big)\,\mathrm{d}s. \tag{13}$$

Proof. Define

$$m(t)=x(0)+\dot{x}(0)+\int_0^t a(s)\big(x(s)+\dot{x}(s)+\ddot{x}(s)\big)\,\mathrm{d}s,\quad m(0)=x(0)+\dot{x}(0), \tag{14}$$

then differentiating (14) and using the fact that $\ddot{x}(t)+x(t)\leqq m(t)$ from (12) we have

$$\dot{m}(t)\leqq a(t)\,(m(t)+\dot{x}(t)). \tag{15}$$

From (12) we observe that $\ddot{x}(t)\leqq m(t)$ for all $t\in I$. Now, integrating $\ddot{x}(t)\leqq m(t)$ from 0 to t we see that the inequality

$$\dot{x}(t)\leqq\dot{x}(0)+\int_0^t m(s)\,\mathrm{d}s, \tag{16}$$

is satisfied for all $t\in I$. Substituting the value of $\dot{x}(t)$ from (16) into (15) we have

$$\dot{m}(t)\leqq a(t)\Big(\dot{x}(0)+m(t)+\int_0^t m(s)\,\mathrm{d}s\Big). \tag{17}$$

Put

$$u(t)=\dot{x}(0)+m(t)+\int_0^t m(s)\,\mathrm{d}s,\quad u(0)=\dot{x}(0)+m(0)=x(0)+2\dot{x}(0), \tag{18}$$

then, differentiating (18) and using the facts that $\dot{m}(t)\leqq a(t)\,u(t)$ and $m(t)\leqq u(t)$ from (17) and (18) we see that the inequality $\dot{u}(t)\leqq[1+a(t)]\,u(t)$, is satisfied, which implies the estimation for $u(t)$ such that

$$u(t)\leqq[x(0)+2\dot{x}(0)]\exp\Big(\int_0^t [1+a(s)]\,\mathrm{d}s\Big).$$

Substituting this value of $u(t)$ in (17) we see that the inequality

$$\dot{m}(t)\leqq a(t)\,[x(0)+2\dot{x}(0)]\exp\Big(\int_0^t [1+a(s)]\,\mathrm{d}s\Big)$$

is satisfied, which implies the estimation for $m(t)$ such that

$$m(t) \leq x(0) + \dot{x}(0) + \int_0^t a(s)[x(0) + 2\dot{x}(0)] \exp\left(\int_0^s [1 + a(\tau)]\, d\tau\right) ds.$$

Now, substituting this value of $m(t)$ in (12) we obtain the desired bound in (13).

We next state and prove the following integrodifferential inequality which can be used in some applications.

Theorem 3. *Let $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$, and $c(t)$ be real-valued nonnegative continuous functions defined on I, for which the inequality*

$$\ddot{x}(t) \leq a + b\left[x(t) + \dot{x}(t) + \int_0^t c(s)\ddot{x}(s)\left(x(s) + \dot{x}(s) + \ddot{x}(s)\right) ds\right], \tag{19}$$

holds for all $t \in I$, where a and b are nonnegative constants.

If $(1+b)[a + b[x(0) + \dot{x}(0)]] \int_0^t c(s) e^{(1+b)s} ds < 1$ *for all $t \in I$, then*

$$\ddot{x}(t) \leq \frac{[a + b[x(0) + \dot{x}(0)]]\, e^{(1+b)t}}{1 - (1+b)[a + b[x(0) + \dot{x}(0)]] \int_0^t c(s) e^{(1+b)s} ds}, \tag{20}$$

for all $t \in I$.

Proof. Define

$$m(t) = a + b\left[x(t) + \dot{x}(t) + \int_0^t c(s)\ddot{x}(s)\left(x(s) + \dot{x}(s) + \ddot{x}(s)\right) ds\right], \tag{21}$$
$$m(0) = a + b[x(0) + \dot{x}(0)],$$

then differentiating (21) and using the fact that $\ddot{x}(t) \leq m(t)$ from (19) together with the facts that $b\dot{x}(t) \leq m(t)$ and $b\left(x(t) + \dot{x}(t)\right) \leq m(t)$ from (21), we see that the inequality

$$\dot{m}(t) \leq (1+b)\, m(t) + (1+b)\, c(t)\, m^2(t),$$

is satisfied, which implies the estimate for $m(t)$ such that (see **[10]**)

$$m(t) \leq \frac{[a + b[x(0) + \dot{x}(0)]]\, e^{(1+b)t}}{1 - (1+b)[a + b[x(0) + \dot{x}(0)]] \int_0^t c(s) e^{(1+b)s} ds},$$

for all $t \in I$. Now, substituting this value of $m(t)$ in (19). We obtain the desired bound in (20).

It is interesting to note that the usefulness of the integrodifferential inequalities established in this section becomes apparent if we consider $x(0)$, $\dot{x}(0)$, $a(t)$, $c(t)$, a and b are known and $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ as unknown, i.e. the inequalities established in this section gives us the bounds in terms of the known functions which majoriges $\ddot{x}(t)$ and consequently $x(t)$ after twice integration. We also note that in the special case when $c(t)=0$, the bound obtained in Theorem 3 can be imporved to a large extent.

3. An Application. In this section, we indicate a simple application of our Theorem 1 to obtain the bound for the solutions of some integrodifferential equations of the form

$$\ddot{x}(t)=f(t)+\int_0^t H[t,\ s,\ x(s),\ \dot{x}(s),\ \ddot{x}(s)]\,\mathrm{d}s, \tag{22}$$

where x, $\dot{x}$, $\ddot{x}$, f, and H are the elements of $\mathbf{R}^n$, the set of real numbers, and continuous on the respective domains of their definitions and $x(0)$ and $\dot{x}(0)$ are given. Let $x(t)\geqq 0$, $\dot{x}(t)\geqq 0$ and $\ddot{x}(t)\geqq 0$ for all $t\in\mathbf{R}$. In Theorem 4 below we obtain a bound for $|\ddot{x}(t)|$, under some suitable conditions on the functions involved in (22).

Theorem 4. *Let the functions f and H in* (22) *sasisfy*

$$|f(t)|\leqq|x(0)|+|\dot{x}(0)|, \tag{23}$$

$$|H[t,\ s,\ x(s),\ \dot{x}(s),\ \ddot{x}(s)]|\leqq a(s)[|x(s)|+|\dot{x}(s)|+|\ddot{x}(s)|], \tag{24}$$

for all t, $s\in I$, where $a(s)$ is a continuous function defined on I. Then

$$|\ddot{x}(t)|\leqq|x(0)|+|\dot{x}(0)|+\int_0^t a(s)\Big[2[|x(0)|+|\dot{x}(0)|]\,\overline{\exp}\Big(\int_0^s a(\tau)\,\mathrm{d}\tau\Big) + \int_0^s [|\dot{x}(0)|+2[|x(0)|+|\dot{x}(0)|]]\exp\Big(\int_0^t [2+a(n)]\,\mathrm{d}n\Big)\exp\Big(\int_\tau^s a(n)\,\mathrm{d}n\Big)\,\mathrm{d}\tau\Big]\,\mathrm{d}s \tag{25}$$

for all $t\in I$.

Proof. Using (23), (24) in (22) we have

$$|\ddot{x}(t)|\leqq|x(0)|+|\dot{x}(0)|+\int_0^t a(s)\Big(|x(s)|+|\dot{x}(s)|+|\ddot{x}(s)|\Big)\,\mathrm{d}s.$$

Now an application of Theorem 1 yields the desired bound in (25).

Finally we note that the integrodifferential inequalities established in this note have many possible applications in the theory of integrodifferential equations. This we propose to consider in a later paper.

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NOTA O INTEGRO-DIFERENCIJALNOJ NEJEDNAKOSTI DRUGOG REDA GRONWAL-BELLIMANOVOG TIPA

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U prvom delu ovog rada izloženi su neki rezultati koji se odnose na integro-diferencijalne nejednakosti Gronwall-Bellmanovog tipa (Teorema 1). U drugom delu date su neke primene rezultata iz prvog dela.