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683. APPLICATIONS OF FUNCTIONAL EQUATIONS TO DIFFERENTIAL EQUATIONS*

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In solving various problems concerning differential equations, particularly when generalisations are attempted, one often arrives at a functional equation. In this note we shall exhibit a few such examples.

1. Certain classes of first order undetermined differential equations in two unknown functions can be solved without integration. For example, the equation

$$\frac{\mathrm{d}}{\mathrm{d}x} M(x,\ u,\ v)+N(x,\ u,\ v)=0 \tag{1.1}$$

reduces to the *algebraic* system for u and v:

$$M(x,\ u,\ v)= -A(x),\quad N(x,\ u,\ v)=A'(x),$$

where A is an arbitrary twice differentiable function. D. S. MITRINOVIĆ [1] reduced some second order equations

$$F(x,\ y,\ y',\ y'';\ z,\ z',\ z'')=0 \tag{1.2}$$

by means of the substitution

$$y=\exp\int u\,\mathrm{d}x,\qquad z=\exp\int v\,\mathrm{d}x \tag{1.3}$$

to the form (1.1).

In [2] we obtained the general form of the equation (1.2) which can be reduced to the form (1.1) by means of (1.3). In solving this problem (the details are given in [2]) we arrived at the following functional equations

$$F(x_1,\ \lambda x_2,\ \lambda x_3,\ \lambda x_4,\ \mu x_5,\ \mu x_6,\ \mu x_7)=f(\lambda)\,g(\mu)\,G(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7);$$

$$H(x_1,\ x_2,\ x_3+x_2^2,\ x_4,\ x_5+x_4^2)=A(x_1,\ x_2,\ x_4)+B(x_1,\ x_2,\ x_4)\,x_3+C(x_1,\ x_2,\ x_4)\,x_5.$$

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Having solved those two equations, the solution of the proposed problem was obtained. In fact, it was shown that the above method can be applied to differential equations of the form

$$\frac{y''}{y} P\left(x, \frac{y'}{y}, \frac{z'}{z}\right) + \frac{z''}{z} Q\left(x, \frac{y'}{y}, \frac{z'}{z}\right) + R\left(x, \frac{y'}{y}, \frac{z'}{z}\right) = 0$$

provided that $\partial P(x_1, x_2, x_3)/\partial x_3 = \partial Q(x_1, x_2, x_3)/\partial x_2$.

2. In [3] we applied the variation of parameters to some nonlinear second order equations of the form

$$y'' + f(x) y' = F(x, y, y'). \tag{2.1}$$

Since the linear part of (2.1), namely $y'' + f(x) y' = 0$, implies $y' = Ke^{-\int f(x)\,dx}$ (K arbitrary constant), the problem is to determine the form of (2.1), i.e. the form of F, so that

$$y' = K(y)\, e^{-\int f(x)\,dx} \tag{2.2}$$

reduces (2.1) to a first order equation. From (2.2) after differentiation and substitution into (2.1) we find

$$K'(y) K(y) e^{-2\int f(x)\,dx} = F\left(x, y, K(y) e^{-\int f(x)\,dx}\right). \tag{2.3}$$

Clearly, (2.3) will be a first order equation in $K(y)$ if

$$F\left(x, y, K(y) e^{-\int f(x)\,dx}\right) = e^{-2\int f(x)\,dx} M(y, K(y)),$$

which leads to the functional equation

$$F\left(z_1, z_2, z_3 g(z_1)\right) = g(z_1)^2 M(z_2, z_3),$$

with the solution

$$F(w_1, w_2, w_3) = g(w_1)^2 M\left(w_2, \frac{w_3}{g(w_1)}\right).$$

Hence, this method can be applied to differential equations (2.1) of the form

$$y'' + f(x) y' = e^{-2\int f(x)\,dx} M\left(y, y' e^{\int f(x)\,dx}\right).$$

REMARK. For $M(s, t) = \sum_{\nu=1}^{n} h_\nu(s)\, t^{\alpha_\nu}$ we obtain the equation

$$y'' + f(x) y' = \sum_{\nu=1}^{n} h_\nu(y) (y')^{\alpha_\nu} e^{(\alpha_\nu - 2)\int f(x)\,dx}$$

considered in [3].

3. Separation of variables is an effective method in the theory of linear partial differential equations. However, it can be applied in various forms, to nonlinear equations, and it will lead to solutions containing a certain number (five at most) of arbitrary constants.

As an example, consider the partial differential equation

$$E(x, y, u, u_x, u_y, u_{xx}, u_{yy})=0 \tag{3.1}$$

and suppose that $u(x, y)=X(x)Y(y)$ is a solution of (3.1). Then

$$E(x, y, XY, X'Y, XY', X''Y, XY'')=0. \tag{3.2}$$

In order that (3.2) separates into two ordinary differential equations it is sufficient that

$$E(z_1, z_2, z_3, z_4, z_5, z_6, z_7)=F_1(z_1, z_3, z_4, z_6)+F_2(z_2, z_3, z_5, z_7),$$

where F_1 and F_2 are generalised homogeneous functions, i.e.

$$F_i(w_1, tw_2, tw_3, tw_4)=\varphi_i(t)f_i(w_1, w_2, w_3, w_4) \qquad (i=1, 2). \tag{3.3}$$

Using the argument given in [4, pp. 304 – 305] we obtain the general solution (excluding the trivial cases) of the equation (3.3) in the form

$$F_i(w_1, w_2, w_3, w_4)=a_i w_2^{k_i} G\left(w_1, \frac{w_3}{w_2}, \frac{w_4}{w_2}\right),$$

$$f_i(w_1, w_2, w_3, w_4)=w_2^{k_i} G_i\left(w_1, \frac{w_3}{w_2}, \frac{w_4}{w_2}\right),$$

$$\varphi_i(w)=a_i w^{k_i}$$

$(i=1, 2)$, where a_i, k_i are arbitrary constants and G_i are arbitrary functions. Hence, the required form of the equation (3.1) is

$$u^{k_1} F_1\left(x, \frac{u_x}{u}, \frac{u_{xx}}{u}\right)+u^{k_2} F_2\left(y, \frac{u_y}{u}, \frac{u_{yy}}{u}\right)=0. \tag{3.4}$$

The equation (3.4) with

$$u(x, y)=X(x)Y(y) \tag{3.5}$$

reduces to two ordinary differential equations

$$X(x)^{k_1-k_2} F_1\left(x, \frac{X'(x)}{X(x)}, \frac{X''(x)}{X(x)}\right)=\lambda, \tag{3.6}$$

$$Y(y)^{k_2-k_1} F_2\left(y, \frac{Y'(y)}{Y(y)}, \frac{Y''(y)}{Y(y)}\right)=-\lambda, \tag{3.7}$$

where λ is arbitrary. Since the general solutions of (3.6) and (3.7) contain two arbitrary constants each, in some cases (3.5) will yield a solution of (3.1) with five arbitrary constants (a complete integral).

Consider again the equation (3.1) and attempt solutions of the form $u(x, y)=X(x)+Y(y)$. We get

$$E(x, y, X+Y, X', Y', X'', Y'')=0. \tag{3.8}$$

If (3.8) is to separate into two ordinary differential equations, we must have

$$E(x, y, X+Y, X', Y', X'', Y'')=H(x, X, X', X'')+K(y, Y, Y', Y''),$$

which leads to the functional equation

$$E(z_1, z_2, z_3+z_4, z_5, z_6, z_7, z_8)=H(z_1, z_3, z_5, z_7)+K(z_2, z_4, z_0, z_8).$$

It is not difficult to see that that the general continuous solution of this equation is given by

$$E(w_1, w_2, w_3, w_4, w_5, w_6, w_7)=F(w_1, w_4, w_6)+G(w_2, w_5, w_7)+cw_3,$$

$$H(w_1, w_2, w_3, w_4)=F(w_1, w_3, w_4)+cw_2,$$

$$K(w_1, w_2, w_3, w_4)=G(w_1, w_3, w_4)+cw_2,$$

where F and G are arbitrary functions and c is an arbitrary constant. Hence, the required form of (3.1) is

$$cu=F(x, u_x, u_{xx})+G(y, u_y, u_{yy}). \tag{3.9}$$

EXAMPLE. The transonic equation

$$u_{xx}=u_y u_{yy} \tag{3.10}$$

belongs to both classes (3.4) and (3.9). If we apply the first method, we arrive at the solution of (3.10) in the form $u(x, y)=X(x)Y(y)$, where X and Y are defined by

$$\int \frac{\mathrm{d}X}{\sqrt{\frac{2\lambda}{3}X^3+A}}=x+B, \qquad \int \frac{\mathrm{d}Y}{\sqrt[3]{\frac{3\lambda}{2}Y^2+C}}=y+D,$$

where λ, A, B, C, D are arbitrary constants.

If we disregard the constants A and C it is easily shown that $u(x, y)=\dfrac{1}{3}\dfrac{(y+D)^3}{(x+B)^2}$ is a solution of (3.10). Moreover, if $u=U$ is a solution of (3.10), then evidently $u=U+Ax+C$ (A, C arbitrary constants) is also a solution of (3.10). Hence, we obtain an explicit solution of (3.10) containing four arbitrary constants:

$$u(x, y)=\frac{1}{3}\frac{(y+D)^3}{(x+B)^2}+Ax+B.$$

The second method leads to the solution

$$u(x, y)=\frac{1}{3\lambda}(2\lambda y+a)^{3/2}+\frac{1}{2}\lambda x^2+bx+c,$$

where λ, a, b, c are arbitrary constants.

4. If u is a solution of a linear equation $Lu=0$, then so is λu, where λ is any scalar. For nonlinear differential equations the following question may be considered: If $Y(x)$ is a solution of the equation

$$y^{(n)}=F(x, y, y', \ldots, y^{(n-1)}) \tag{4.1}$$

do there exist constants λ and μ such that $y(x)=\lambda Y(\mu x)=\lambda Y(X)$ is also a solution of (4.1)? If such constants exist, they are called constants of superposition [5].

In solving the above problem we shall always arrive at a functional equation. As an example consider the second order equation ([5])

$$y''=F(x, y, y'). \tag{4.2}$$

If this equation is to be invariant under the two parameter group $y=\lambda Y$, $x=\mu^{-1}X$, F would have to satisfy the functional equation

$$F(\mu^{-1}z_1, \lambda z_2, \lambda\mu z_3)=\lambda\mu^2 F(z_1, z_2, z_3), \tag{4.3}$$

AMES [5] solved this equation by assuming that $F\in C^1$, and by reducing it to partial differential equations. In fact, employing the technique described in [4], it is easily shown that the general continuous solution (excluding trivial cases) of (4.3) is given by $F(w_1, w_2, w_3)=w_2 w_1^{-2} A(w_1 w_2^{-1} w_3)$, where A is an arbitrary function. This leads to the following form of (4.2): $y''=yx^{-2}A\left(\frac{xy'}{y}\right)$.

In certain cases λ is a function of μ. Suppose $\lambda=\mu^k$. The functional equation (4.3) becomes

$$F(\mu^{-1}z_1, \mu^k z_2, \mu^{k+1}z_3)=\mu^{k+2}F(z_1, z_2, z_3)$$

with the solution

$$F(w_1, w_2, w_3)=w_1^{-k-2}A(w_1 w_2^{1/k}, w_2^{-1/k}w_3^{1/(k+1)}),$$

where A is an arbitrary function. Hence, in this case the equation (4.2) takes the form

$$y''=x^{-k-2}A(xy^{1/k}, y^{-1/k}(y')^{1/(k+1)}). \tag{4.4}$$

In particular, if $A(s, t)=-s^{k+2}-2s^{k+1}t^{k+1}$, (4.4) becomes the EMDEN equation

$$y''+\frac{2}{x}y'+y^n=0 \quad \left(n=\frac{k+2}{k}\right) \tag{4.5}$$

with the well known property that if $Y(x)$ is a solution of (4.5), then so is $C^k Y(Cx)$, i.e. $C^{2/(n-1)}Y(Cx)$ for arbitrary C.

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PRIMENE FUNKCIONALNIH JEDNAČINA U DIFERENCIJALNIM JEDNAČINAMA

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Prilikom rešavanja raznih problema u vezi sa diferencijalnim jednačinama. a naročito prilikom generalizacija, često se dolazi do funkcionalnih jednačina. U ovom radu dato je nekoliko takvih primera.

U prvom odeljku traži se oblik neodređene diferencijalne jednačine drugog reda na čiju se integraciju može primeniti metod D. S. Mitrinovića. U drugom odeljku traži se oblik funkcije F u (2.1) tako da se na tu jednačinu može primeniti jedna varijacija metoda varijacije konstanata. Treći odeljak odnosi se na parcijalne jednačine drugog reda. Traži se oblik jednačine koja dopušta rešavanje multiplikativnim i aditivnim razdvajanjem promenljivih. Najzad, u četvrtom odeljku određuje se diferencijalna jednačina drugog reda koja je invarijantna u odnosu na dvoparametarsku grupu $y=\lambda Y$, $x=\mu^{-1}X$.

Prilikom rešavanja navedenih problema uvek se dolazi do funkcionalnih jednačina. Rešavanjem dobijenih funkionalnih jednačina rešeni su i postavljeni problemi.