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678. **ON AN INEQUALITY***

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0. Noting that the inequality

$$\min\left((a-b)^2,\ (b-c)^2,\ (c-a)^2\right) \leqq \frac{1}{2}(a^2+b^2+c^2) \tag{1}$$

holds for different real numbers, D. S. MITRINOVIĆ in the American Mathematical Monthly, vol. 74 (1967), p. 1134, proposed problem E 2032: to determine an upper bound for the expression

$$\min_{1\leqq k<i\leqq n} (a_k-a_i)^2,$$

where $a_1, \ldots, a_n$ $(n>1)$ are different real numbers.

In the same journal, vol. 75 (1968), p. 1124, a solution by J. LEHNER was published, and it was stated that this problem was also solved by 21 mathematicians.

In vol. 76 (1969), pp. 691 — 692 of this journal, a correct solution by M. NEWMAN and J. LEHNER was published, since the original solution presented by J. LEHNER was found to be incorrect. It is interesting that out of 21 solutions an incorrect solution was published. That is why MITRINOVIĆ turned directly to some other solvers of the problem: D. D. ADAMOVIĆ (Yugoslavia), E. S. LANGFORD (USA), D. C. B. MARSH (USA), J. POLAJNAR (Yugoslavia) and asked them for their solutions, which they willingly placed at his disposal.

1. We present at first solution by NEWMAN and LEHNER.

If $a_1, \ldots, a_n$ are real numbers, then

$$\min_{a_i\neq a_j} (a_i-a_j)^2 = \mu^2 (a_1{}^2+\cdots+a_n{}^2), \qquad \mu^2=\frac{12}{n\,(n^2-1)}.$$

Proof. We may assume $a_1 \leqq \cdots \leqq a_n$ and, by homogeneity, $\sum_{i=1}^{n} a_i{}^2=1$. Now

$$\sum_{1\leqq i<j\leqq n} (a_i-a_j)^2 = n\sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i\right)^2. \tag{*}$$

* Presented May 6, 1980 by A. LUPAŞ, and June 23, 1980 by D. C. B. MARSH.

Assume $a_{i+1}-a_i>\mu>0$, $i=1, \ldots, n-1$. Then $(a_j-a_i)^2>(i-j)^2\mu^2$, $i, j=1, \ldots, n$ and

$$\sum_{1\leq i<j\leq n}(a_i-a_j)^2>\mu^2\cdot\sum_{1\leq i<j\leq n}(i-j)^2=\mu^2\cdot\frac{n^2(n^2-1)}{12}=n.$$

Inserting this in (*) we get

$$n<n\sum_{i=1}^{n}a_i^2-\left(\sum_{i=1}^{n}a_i\right)^2\leq n\sum_{i=1}^{n}a_i^2,$$

or $\sum_{i=1}^{n}a_i^2>1$, a contradiction to our normalization. Hence $\min_{a_i\neq a_j}(a_i-a_j)^2\leq\mu^2$, as asserted. For each n the inequality is sharp. □

2. The solution by POLAJNAR reads:

We hawe

$$\min_{1\leq k<i\leq n}(a_k-a_i)^2\leq\frac{12}{n(n^2-1)}\sum_{i=1}^{n}a_i^2. \tag{2}$$

Proof. Let $a_1<\cdots<a_n$, $\min_{1\leq k<i\leq n}(a_k-a_i)^2=d^2\ (d>0)$, $\min_i a_i^2=a_j^2$, where j is a fixed index. Then, for $i>j$, $a_i>0$, while for $i<j$, $a_i<0$.

For $i=1, \ldots, n$, put $b_i=a_j+(i-j)d$ which implies $b_i=b_1+(i-1)d$.

Then, for $i>j$,

$$a_i=(a_i-a_{i-1})+\cdots+(a_{j+1}-a_j)+a_j\geq(i-j)d+a_j=b_i\geq a_j\geq -a_i, \tag{3}$$

and similarly, for $i<j$,

$$a_i\leq b_i\leq -a_i. \tag{4}$$

From (2) and (3) we have

$$a_i^2\geq b_i^2,$$

i.e.,

$$\sum_{i=1}^{n}a_i^2\geq\sum_{i=1}^{n}b_i^2=\sum_{i=1}^{n}(b_1^2+2b_1d(i-1)+(i-1)^2d^2).$$

Whence, by the formula for the sum of squares of natural numbers, we have

$$\sum_{i=1}^{n}a_i^2\geq\sum_{i=1}^{n}b_i^2=n\left(b_1+\frac{n-1}{2}d\right)^2+\frac{n(n^2-1)}{12}d^2\geq\frac{n(n^2-1)}{12}d^2,$$

which implies inequality (2). □

COMMENT BY S. B. PREŠIĆ. From (2) it follows that the inequality

$$\min_{1\leq k<i\leq n}(a_k-a_i)^2\leq\frac{12}{n(n^2-1)}\sum_{i=1}^{n}(a_i+t)^2$$

holds tor any real number t.

Since the function f defined by $f(t)=\sum_{i=1}^{n}(a_i+t)^2$ attains its minimum

$$\sum_{i=1}^{n} a_i^2-\frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2 \text{ for } t=-\frac{1}{n}\sum_{i=1}^{n} a_i,$$

we get the inequality

$$\min_{1\leqq k<i\leqq n}(a_k-a_i)^2\leqq\frac{12}{n(n^2-1)}\left(\sum_{i=1}^{n} a_i^2-\frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2\right),$$

which is stronger than (1). □

3. The solution by MARSH reads:

$$\min_{1\leqq k<i\leqq n}(a_k-a_i)^2\leqq\frac{4^{1/n}}{(n-1)\left((n-1)!\prod_{k=1}^{n-1}k!\prod_{k=2}^{n-1}(k-1)^{n-k}\right)^{2/n(n-1)}}\sum_{i=1}^{n} a_i^2. \tag{5}$$

Proof. Let $a_1<\cdots<a_n$ and define $d_j=a_{j+1}-a_j$ $(j=1,\ldots,n-1)$, $d_0=0$.

Then, the expression

$$\sum_{i=1}^{n} a_i^2=\sum_{i=0}^{n-1}\left(a_1+\sum_{k=0}^{i} d_k\right)^2$$

is minimized for $a_1=-\frac{1}{n}\sum_{k=0}^{n-1}(n-k)\,d_k$.

Therefore, we have

$$\sum_{i=1}^{n} a_i^2\geqq\frac{1}{n^2}\sum_{1\leqq i\leqq j\leqq n} c_{ij}\,d_i\,d_j.$$

Using the fact that the geometric mean of a set of positive numbers is no greater than their arithmetic mean, we may next write

$$\sum_{i=1}^{n} a_i^2\geqq\frac{1}{n^2}C^{1/t}\,t\left(\prod_{i=1}^{n-1} d_i^n\right)^{1/t}\geqq\frac{t}{n^2}C^{1/t}\min_i d_i^2\quad\left(t=\binom{n}{2},\; C=\prod c_{ij}\right).$$

It is evident that $\min_i d_i^2=\min_{1\leqq k<i\leqq n}(a_k-a_i)^2$, so that we can write

$$\min_{1\leqq k<i\leqq n}(a_k-a_i)^2\leqq\gamma_n\left(\sum_{i=1}^{n} a_i^2\right),$$

where, explicitely,

$$\gamma_n=\frac{4^{1/n}}{(n-1)\left((n-1)!\prod_{k=1}^{n-1}k!\prod_{k=2}^{n-1}(k-1)^{n-k}\right)^{2/n(n-1)}}.\ \square$$

4. The solution by ADAMOVIĆ and LANGFORD, which they arrived at independently of each other reads:

$$\min_{1\leqq k<i\leqq n}(a_k-a_i)^2\leqq\frac{2}{(n-1)^2}\left((\min a_k)^2+(\max a_k)^2\right). \tag{6}$$

Proof. Let $a_1 < \cdots < a_n$ and $D = \min_{1 \leqq k \leqq n-1} (a_{k+1} - a_k)$. Then

$$D \leqq \frac{1}{n-1} \sum_{k=1}^{n-1} (a_{k+1} - a_k) = \frac{1}{n-1} (a_n - a_1),$$

and consequently,

$$\min_{1 \leqq k < i \leqq n} (a_k - a_i)^2 = D^2 \leqq \frac{1}{(n-1)^2} (a_n - a_1)^2 \leqq \frac{1}{(n-1)^2} (|a_1| + |a_n|)^2$$

$$\leqq \frac{2}{(n-1)^2} (a_1^2 + a_n^2),$$

which is just inequality (6).

Inequality (6) implies the following generalization of (1):

$$\min_{1 \leqq k < i \leqq n} (a_k - a_i)^2 \leqq \frac{2}{(n-1)^2} \sum_{i=1}^{n} a_i^2. \tag{7}$$

Equality holds in (7), under the assumption $a_1 \leqq \cdots \leqq a_n$, if and only if

$a_1 + a_2 = 0$, when $n = 2$,

$a_1 + a_3 = a_2 = 0$, when $n = 3$,

$a_i = 0 \, (1 \leqq i \leqq n)$, when $n > 3$. □

REMARK 1. Notice that inequalities (2), (5) and (7) for $n = 3$ reduce to (1), whereas inequality (6) is sharper than (1). For arbitrary n, inequalities (2), (5) and (7) cannot be compared to one another.

5. In the journal of the College of Arts and Sciences — Chiba University, vol. 5 (1968), No. 2, pp. 199 — 203, N. OZEKI published a paper in Japanese with the title: *On the estimation of inequalities by the maximum or minimum values*, but without any summary in another language.

In his paper OZEKI has, among other things, made without proof the following statement:

If $a_1, \ldots, a_n$ are different real numbers, $d = \min_{i \neq j} |a_i - a_j|$, and if $p > 0$ is a given real number, then

$$\sum_{i=1}^{n} |a_i|^p \geqq C_p d^p, \tag{8}$$

where

$$C_p = 2(1^p + 2^p + \cdots + m^p), \qquad n = 2m+1, \tag{9}$$

$$= 2^{1-p} (1^p + 3^p + \cdots + (2m-1)^p), \qquad n = 2m.$$

Proof. For a given $d>0$ denote by $S=S(d)$ the set of all ordered n-tuples $a=(a_1, \dots, a_n)$ of real numbers subject to $d=\min_{i\neq j}|a_i-a_j|$, and put

$$f(a)=\sum_{i=1}^{n}|a_i|^p \qquad (a\in S). \tag{10}$$

Suppose first that $d=1$. Then the statement reduces to $f(a)\geqq C_p\,(a\in S)$. Since $f(a)$ does not depend on the order of the summands, we may suppose that $a_1<a_2<\dots<a_n$. Put $c=\min|a_i|$, and let, for instance, $c=|a_k|$. We may also suppose that $a_k\geqq 0$, i.e. $c=a_k$.

If $a_k>1$, then $b=(a_1, \dots, a_{k-1}, 1, a_{k+1}, \dots, a_n)\in S$ and $f(a)\geqq f(b)$. Thus we only consider the case $a_k\leqq 1$, which implies $0\leqq c\leqq 1$. Since $d=a_{i+1}-a_i$ $(1\leqq i<n)$ we have

$$a_{k+r}\geqq c+r\,(>0) \qquad (0\leqq r\leqq n-k),$$

$$a_{k-s}\leqq c-s\,(\leqq 0) \qquad (1\leqq s\leqq k-1),$$

and (10) yields

$$f(a)\geqq \sum_{s=1}^{k-1}(s-c)^p+\sum_{r=0}^{n-k}(r+c)^p. \tag{11}$$

Put $n=2m$, or $n=2m+1$, depending on whether n is even or odd. Then $k-1\geqq m$ or $n-k\geqq m$. Suppose, for example, that $k-1\geqq m$, and hence $m\geqq n-k$. Since $s-c\geqq r+c$ for $s>r\geqq 1$, according to (11) we have

$$f(a)\geqq \sum_{s=1}^{m}(s-c)^p+\sum_{r=0}^{\hat{m}}(r+c)^p, \tag{12}$$

where $\hat{m}=m-1$ for $n=2m$, and $\hat{m}=m$ for $n=2m+1$. Denote by $h(c)$ $(0\leqq c\leqq 1)$ the right hand side of (12). Since $h''(c)>0$, the function h' is increasing on $[0, 1]$. For $\hat{m}=m$ we have $h'(0)=0$, and hence $h'(c)\geqq 0$ $(0\leqq c\leqq 1)$, implying $f(a)\geqq h(c)\geqq h(0)=C_p$ for $n=2m+1$. If $\hat{m}=m-1$, then $h'(1/2)=0$, and so $f(a)\geqq h(c)\geqq h(1/2)=C_p$ for $n=2m$.

This proves the statement for $d=1$. If $d\neq 1$, dividing (8) by d^p, we see that it is equivalent to $f(b)\geqq C_p$, where $b_i=a_i/d$ $(1\leqq i\leqq n)$. Since, in addition, $\min_{i\neq j}|b_i-b_j|=1$, the statement is reduced to the preceeding case.

Finally, since equality holds in (8) for $n=2m+1$ and $-a_i=a_{m+i}=i$ $(1\leqq i\leqq m)$, $a_m=0$, and for $n=2m$ and $-a_i=a_{m+i}=\frac{1}{2}+i$ $(0<i\leqq m)$, we conclude that C_p defined by (9) is the greatest real number such that (8) holds. □

6. Suppose that $p=1$. For n odd, we have

$$C_1=2\left(1+2+\dots+\frac{n-1}{2}\right)=\frac{1}{4}(n^2-1).$$

If n is even, then

$$C_1 = 1 + 3 + \cdots + (n-1) = \frac{n^2}{4}.$$

Suppose that $p=2$. If n is odd, then

$$C_2 = 2\left(1^2 + 2^2 + \cdots + \left(\frac{n-1}{2}\right)^2\right) = \frac{n(n^2-1)}{12},$$

where the following formula was applied:

$$1^2 + 2^2 + \cdots + k^2 = \frac{1}{6} k(k+1)(2k+1).$$

If n is even, then

$$C_2 = \frac{1}{2}\left(1^2 + 3^2 + \cdots + (n-1)^2\right) = \frac{1}{12} n(n^2-1).$$

Thus, for every natural number n from (8) and (9) we hawe deduced the particular case treated by NEWMAN-LEHNER, POLAJNAR, MARSH, ADAMOVIĆ--LANGFORD.

For $p=3$ and n odd we have

$$C_3 = 2\left(1^3 + 2^3 + \cdots + \left(\frac{n-1}{2}\right)^3\right) = \frac{1}{32}(n^2-1)^2,$$

since

$$1^3 + 2^3 + \cdots + k^3 = \frac{1}{4} k^2 (k+1)^2.$$

For $p=3$ and n even we have

$$C_3 = \frac{1}{4}(1^3 + 3^3 + \cdots + (n-1)^3) = \frac{1}{32} n^2 (n^2-2).$$

Let $p=4$. Then, for n even,

$$C_4 = \frac{1}{8}(1^4 + 3^4 + \cdots + (n-1)^4).$$

Since

$$1^4 + 3^4 + \cdots + (2k+1)^4 = \frac{1}{15}(k+1)(2k+1)(2k+3)(12k^2 + 24k + 5),$$

we obtain

$$C_4 = \frac{1}{240} n(n^2-1)(3n^2-4).$$

If n is odd, then

$$C_4 = 2\left(1^4 + 2^4 + \cdots + \left(\frac{n-1}{2}\right)^4\right).$$

Since

$$1^4+2^4+\cdots+k^4=\frac{1}{30}k(k+1)(2k+1)(3k^2+3k-1),$$

we have

$$C_4=\frac{1}{240}n(n^2-1)(3n^2-4).$$

Thus we conclude that

$$\min_{1\leq i<j\leq n}|a_i-a_j|^4\leq\frac{240}{n(n^2-1)(3n^2-4)}\sum_{i=1}^{n}|a_i|^4.$$

REMARK 2. The constants C_1 and C_3 may be written under the form

$$C_1=\left[\frac{n}{2}\right]\cdot\left[\frac{n+1}{2}\right],\quad n=1,\ 2,\ldots.$$

$$C_3=\frac{1}{32}\left(n^4-2n^2+n-2\left[\frac{n}{2}\right]\right),\quad n=1,\ 2,\ldots.$$

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O JEDNOJ NEJEDNAKOSTI

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U ovom članku pisci daju dokaz nejednakosti (8), gde je C_p definisano pomoću (9). Takođe je izložen istorijat jednog MITRINOVIĆevog problema koji je doveo do nejednakosti (8).