UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 634 — № 677 (1979), 221—227.

672. ON THE UNIVALENCE OF RATIONAL FUNCTIONS*

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1. Consider the function

$$z \mapsto f(z) = \frac{z}{(1+z^n)^k} \qquad (k,\ n = 1,\ 2, \ldots), \tag{1}$$

and suppose that $nk-1>0$, which excludes the case $f(z)=\frac{z}{1+z}$, whose domain of univalence is the whole z-plane.

The function f is regular in the disk $|z|<1$. Let z_1 and $z_2\ (z_1 \neq z_2)$ be arbitrary points of the disk $|z|<r\ (r \leqq 1)$, i.e. let $|z_1|<r$ and $|z_2|<r$, and start with the difference

$$f(z_1)-f(z_2)=\frac{z_1(1+z_2{}^n)^k-z_2(1+z_1{}^n)^k}{(1+z_1{}^n)^k(1+z_2{}^n)^k}.$$

By a repeated use of the inequalities

$$|a|-|b| \leqq |a+b| \leqq |a|+|b|$$

we get

$$|f(z_1)-f(z_2)|>|z_1-z_2|\frac{A}{(1+r^n)^{2k}},$$

where

$$A \stackrel{\text{def}}{=} 1-\binom{k}{1}(n-1)r^n-\binom{k}{2}(2n-1)r^{2n}-\cdots-\binom{k}{k}(kn-1)r^{kn}$$
$$\equiv \bigl(1-(nk-1)r^n\bigr)(1+r^n)^{k-1}.$$

If $A>0$, which is fulfilled for $r<1/\sqrt[n]{nk-1}$, the implication

$$z_1 \neq z_2 \ \Rightarrow\ f(z_1) \neq f(z_2)$$

is valid.

Since the zeroes $z_p\ (p=1, \ldots, n)$ of the function f' are such that $|z_p|=1/\sqrt[n]{nk-1}$, we arrive at the result:

Theorem 1. *The function* $z \mapsto \frac{z}{(1+z^n)^k}$ $(n,\ k=1,\ 2, \ldots)$ *is univalent in the disk* $|z|<r$, *with the maximal radius* r *given by* $\frac{1}{\sqrt[n]{nk-1}}$.

* Received February 1, 1980. Presented by Professor I. E. BAZILEVIČ (Moscow).

REMARK. The function $z \mapsto \frac{z}{(1+az^n)^k}$, by means of the substitution $z\sqrt[n]{a}=t$, reduces to the function $t \mapsto \frac{t/\sqrt[n]{a}}{(1+t^n)^k}$ which we already considered.

2. Consider now the function

$$z \mapsto f(z) = \frac{z}{1+a_1 z+\cdots+a_k z^k} \qquad (a_k \neq 0), \tag{2}$$

which contains as a particular case the function (1).

Using the same procedure as in Section **1**, we find

$$f(z_1)-f(z_2)=(z_1-z_2)\frac{1-z_1 z_2\left(a_2+a_3(z_1+z_2)+\cdots+a_k(z_1^{k-2}+z_1^{k-3}z_2+\cdots+z_2^{k-2})\right)}{(1+a_1 z_1+\cdots+a_k z_1^k)(1+a_1 z_2+\cdots+a_k z_2^k)}, \tag{3}$$

$$|f(z_1)-f(z_2)|>|z_1-z_2|\frac{1-r^2\left(|a_2|+2r|a_3|+\cdots+(k-1)|a_k|r^{k-2}\right)}{(1+|a_1|r+\cdots+|a_k|r^k)^2}. \tag{4}$$

Here again z_1 and z_2 denote arbitrary points of $|z|<r$, where r should be chosen so that the polynomial $P(z)=1+a_1 z+\cdots+a_k z^k$ has no zeroes in the disk $|z|<r$.

If

$$1-|a_2|r^2-2|a_3|r^3-\cdots-(k-1)|a_k|r^k>0, \tag{5}$$

then the expression on the right hand side of (4) is positive.

Hence, the implication $z_1 \neq z_2 \Rightarrow f(z_1)\neq f(z_2)$ is valid.

Since

$$f'(z)=\frac{1-a_2 z^2-2a_3 z^3-\cdots-(k-1)a_k z^k}{(1+a_1 z+\cdots+a_k z^k)^2},$$

the zeroes of the function f' are given by the equation

$$(k-1)a_k z^k+\cdots+2a_3 z^3+a_2 z^2-1=0. \tag{6}$$

In order to determine the maximal radius of univalence of the function (2), it is necessary to know suitable informations about roots of equations (6) and

$$1-|a_2|r^2-2|a_3|r^3-\cdots-(k-1)|a_k|r^k=0. \tag{7}$$

This equation has exactly one positive root, which we denote by r_0. If

$$|a_1|\leqq|a_3|r_0^2+2|a_4|r_0^3+\cdots+(k-2)|a_k|r_0^{k-1}, \tag{8}$$

the polynomial P has no zeroes in the disk $|z|<r_0$ because, for $|z|<r_0$,

$$\begin{aligned}|P(z)| &\geqq 1-|a_1||z|-|a_2||z|^2-\cdots-|a_k||z|^k\\ &>1-|a_1|r_0-|a_2|r_0^2-\cdots-|a_k|r_0^k\\ &\geqq 1-|a_2|r_0^2-2|a_3|r_0^3-\cdots-(k-1)|a_k|r_0^k.\end{aligned}$$

If $a_2,\ldots,a_k$ are real nonnegative numbers, then r_0 is a root of the equation (6), too. On the basis of previous considerations one can conclude the following:

Theorem 2. *If the condition* (8) *is satisfied, the unique positive root of* (7) *is a radius of univalence of the function* (2). *If, in addition,* $a_2, \ldots, a_k \geqq 0$, *then* r_0 *is the maximal radius of univalence of the function* (2).

If $a_1, \ldots, a_k$ are positive numbers, then the equation (6) has only one positive root which is, at the same time, a positive root of (7).

A special case of the function (2) is

$$f(z)=\frac{z}{1+z+z^2}; \tag{9}$$

the equations (5) and (6) read: $r^2-1=0$ and $z=\pm 1$ respectively, which implies $r=1$ (the other root is discarded) and $z=\pm 1$ (in these points the function f is not univalent). The function f has two poles $z=\frac{1}{2}(-1\pm i\sqrt{3})$ which lie on the circle $|z|=1$.

Hence, the maximal radius of univalence of the function f, given by (9), is $r=1$.

3. In MARDEN's monograph ([1], p. 126, exercise 2) the following theorem is given:

All the zeroes of polynomial $c_0+c_1z+\cdots+c_kz^k\ (c_0\neq 0)$ *lie on or outside the circle*

$$|z|=\min_{p=1,\ldots,k}\frac{|c_0|}{|c_0|+|c_p|}.$$

According to this theorem, all the roots of the equtions (5) and (6) lie in the region $|z|\geqq r$, where

$$r=\min\left(\frac{1}{1+|a_2|}, \frac{1}{1+2|a_3|}, \ldots, \frac{1}{1+(k-1)|a_k|}\right). \tag{10}$$

If $P(z)$ has no zeroes in the disk $|z|<r$, a radius of univalence of the function (2) is given by (10), but that r need not be the maximal radius.

Apply this theorem to the function

$$z\mapsto f(z)=\frac{z}{1+z+z^2+\cdots+z^k}.$$

The zeroes of the polynomial $P(z)=1+z+z^2+\cdots+z^k$ are given by

$$z_n=e^{\frac{2n\pi i}{k+1}} \qquad (n=1,\ldots,k)$$

and they all lie on the circle $|z|=1$.

The equations (5) and (6) in this case read

$$1-r^2-2r^3-\cdots-(k-1)r^k=0,$$

$$1-z^2-2z^3-\cdots-(k-1)z^k=0,$$

respectively. Applying the mentioned theorem, we get

$$r=\min\left(\frac{1}{1+0}, \frac{1}{1+2}, \ldots, \frac{1}{1+(k-1)}\right)=\frac{1}{k}.$$

Hence, the function f is univalent in the disk $|z|<\frac{1}{k}$, but the radius of univalence is not maximal. After all, we applied a theorem which does not give the best possible result.

4. If we apply the procedure from Section **1** and the theorem from Section **3** to the function f given by

$$f(z)=\frac{b_0+b_1 z+b_2 z^2}{a_0+a_1 z+a_2 z^2}, \tag{11}$$

we arrive at the result:

If $a_0 b_1-a_1 b_0\neq 0$ and $a_0\neq 0$, the function (11) is univalent in the disk $|z|<\rho=\min(r_1, r_2)$, where

$$r_1=\min\left(\frac{|a_0|}{|a_0|+|a_1|},\ \frac{|a_0|}{|a_0|+|a_2|}\right),$$

$$r_2=\min\left(\frac{|a_0 b_1-a_1 b_0|}{|a_0 b_1-a_1 b_0|+2|a_0 b_2-a_2 b_0|},\ \frac{|a_0 b_1-a_1 b_0|}{|a_0 b_1-a_1 b_0|+|a_1 b_2-a_2 b_1|}\right).$$

We cannot claim that ρ is the maximal radius of univalence.

5. Consider now the function f given by

$$f(z)=\frac{b_0+b_1 z+b_2 z^2+b_3 z^3}{a_0+a_1 z+a_2 z^2+a_3 z^3}. \tag{12}$$

The equations which correspond to the equations (5) and (6) in this case read

$$|(a_0 b_1)|-2|(a_0 b_2)|r-(3|(a_0 b_3)|+|(a_1 b_2)|)r^2 -2|(a_1 b_3)|r^3-|(a_2 b_3)|r^4=0, \tag{13}$$

$$(a_0 b_1)+2(a_0 b_2)z+\big(3(a_0 b_3)+(a_1 b_2)\big)z^2+2(a_1 b_3)z^3+(a_2 b_3)z^4=0, \tag{14}$$

where we define

$$(a_i b_j)=a_i b_j-a_j b_i \qquad (i=0,\ 1,\ 2;\ j=1,\ 2,\ 3).$$

To the polynomial $a_0+a_1 z+a_2 z^2+a_3 z^3$ corresponds:

$$r_1=\min\left(\frac{|a_0|}{|a_0|+|a_1|},\ \frac{|a_0|}{|a_0|+|a_2|},\ \frac{|a_0|}{|a_0|+|a_3|}\right) \qquad (a_0\neq 0);$$

and to the polynomial of equation (13) corresponds:

$$r_2=\min\left(\frac{|(a_0 b_1)|}{|(a_0 b_1)|+2|(a_0 b_2)|},\ \frac{|(a_0 b_1)|}{|(a_0 b_1)|+3\big(|(a_0 b_3)|+|(a_1 b_2)|\big)},\right.$$
$$\left.\frac{|(a_0 b_1)|}{|(a_0 b_1)|+2|(a_1 b_3)|},\ \frac{|(a_0 b_1)|}{|(a_0 b_1)|+|(a_2 b_3)|}\right), \qquad (a_0 b_1)\neq 0.$$

Since

$$|3(a_0 b_3)+(a_1 b_2)|\leqq 3|(a_0 b_3)|+|(a_1 b_2)|,$$

r_2 also corresponds to the polynomial of equation (14).

The function (12) is univalent in the disk

$$|z|<\rho=\min(r_1, r_2).$$

6. Consider now the most general rational function

$$(15) \qquad z\mapsto f(z)=\frac{\sum_{i=0}^{m} b_i z^i}{\sum_{i=0}^{m} a_i z^i} \qquad (a_0 b_1 - a_1 b_0 \neq 0 \text{ and } a_0 \neq 0),$$

which contains the case when the polynomials in the numerator and denominator are not of the same degree, since it is enough to take $b_m=0$, $b_{m-1}=0, \ldots, b_{m-q}=0\ (q<m)$ or $a_m=0$, $a_{m-1}=0, \ldots, a_{m-p}=0\ (p<m)$. Formulas become symmetrical when the rational function f is considered in the form (15).

We suppose that the polynomials in the numerator and denominator have no common zeroes.

Let z_1 and z_2 be the points from the disk $|z|<r$ ($|z_1|<r$ and $|z_2|<r$) and consider difference

$$(16) \qquad f(z_1)-f(z_2)=\frac{\sum_{i=0}^{m} b_i z_1^i}{\sum_{i=0}^{m} a_i z_1^i}-\frac{\sum_{i=0}^{m} b_i z_2^i}{\sum_{i=0}^{m} a_i z_2^i}=\frac{\sum_{i=0}^{m} b_i z_1^i \cdot \sum_{i=0}^{m} a_i z_2^i - \sum_{i=0}^{m} b_i z_2^i \cdot \sum_{i=0}^{m} a_i z_1^i}{\sum_{i=0}^{m} a_i z_1^i \cdot \sum_{i=0}^{m} a_i z_2^i}.$$

V. Kocić showed (private communication) that

$$(\mathrm{S}) \qquad \sum_{i=0}^{m} b_i z_1^i \cdot \sum_{i=0}^{m} a_i z_2^i - \sum_{i=0}^{m} b_i z_2^i \cdot \sum_{i=0}^{m} a_i z_1^i = \sum_{\substack{j,i=0\\ i>j}}^{m} (a_j b_i) z_1^j z_2^j (z_1^{i-j} - z_2^{i-j}),$$

where $(a_j b_i)=a_j b_i - a_i b_j$.

We shall now transform the sum on the right-hand side of (S) in the following way

$$A \overset{\text{def}}{=} \sum_{\substack{i,j=0\\ i>j}}^{m} (a_j b_i) z_1^j z_2^j (z_1^{i-j} - z_2^{i-j})$$

$$=(a_0 b_1)(z_1-z_2)+\sum_{i=2}^{m}(a_0 b_i)(z_1^i - z_2^i)+\sum_{\substack{j,i=1\\ i>j}}^{m}(a_j b_i) z_1^j z_2^j (z_1^{i-j}-z_2^{i-j})$$

$$=(z_1-z_2)\Big((a_0 b_1)+\sum_{i=2}^{m}(a_0 b_i)(z_1^{i-1}+z_1^{i-2} z_2+\cdots+z_2^{i-1})\Big)$$

$$+\sum_{\substack{j,i=1\\ i>j}}^{m}(a_j b_i) z_1^j z_2^j (z_1^{i-j-1}+z_1^{i-j-2} z_2+\cdots+z_2^{i-j-1}).$$

We have for A the following estimates

$$|A| \geqq |z_1 - z_2| \Big(|(a_0 b_1)| - \sum_{i=2}^{m} |(a_0 b_i)| (|z_1|^{i-1} + |z_1|^{i-2} |z_2| + \cdots + |z_2|^{i-1})$$

$$- \sum_{\substack{j,i=1\\ i>j}}^{m} |(a_j b_i)| \, |z_1|^j |z_2|^j (|z_1|^{i-j-1} + |z_1|^{i-j-2} |z_2| + \cdots + |z_2|^{i-j-1}) \Big)$$

$$> |z_1 - z_2| \Big(|(a_0 b_1)| - \sum_{i=2}^{m} i \, |(a_0 b_i)| \, r^{i-1} - \sum_{\substack{j,i=1\\ i>j}}^{m} (i-j) \, |(a_j b_i)| \, r^{i+j-1} \Big).$$

Let

$$B \overset{\text{def}}{=} |(a_0 b_1)| - \sum_{i=2}^{m} i \, |(a_0 b_i)| \, r^{i-1} - \sum_{\substack{j,i=1\\ i>j}}^{m} (i-j) \, |(a_j b_i)| \, r^{i+j-1}.$$

If $B > 0$ for sufficiently small r, we have

$$|f(z_1) - f(z_2)| > \frac{|z_1 - z_2| \, B}{(|a_0| + |a_1| \, r + \cdots + |a_m| \, r^m)^2},$$

which yields the implication $z_1 \neq z_2 \Rightarrow f(z_1) \neq f(z_2)$.

The derivative f' is given by

$$f'(z) = \frac{1}{\Big(\sum_{i=0}^{m} a_i z^i\Big)^2} \Big(\sum_{i=1}^{m} i b_i z^{i-1} \cdot \sum_{i=0}^{m} a_i z^i - \sum_{i=0}^{m} b_i z^i \cdot \sum_{i=1}^{m} i a_i z^{i-1} \Big)$$

$$= \frac{1}{\Big(\sum_{i=0}^{m} a_i z^i\Big)^2} \Big((a_0 b_1) + \sum_{i=2}^{m} i (a_0 b_i) z^{i-1} + \sum_{\substack{j,i=1\\ i>j}}^{m} (i-j) (a_j b_i) z^{i+j-1} \Big).$$

The above formula was deduced in the following way. By considering particular cases it was noticed that the polynomial B and the polynomial

$$C \overset{\text{def}}{=} (a_0 b_1) + \sum_{i=2}^{m} i (a_0 b_i) z^{i-1} + \sum_{\substack{j,i=1\\ i>j}}^{m} (i-j) (a_j b_i) z^{i+j-1}$$

are of such structure that, starting with B it is possible to form C. This hypothesis was then proved in the general case by mathematical induction.

Hence, the three equations $\sum_{i=0}^{m} a_i z^i = 0$, $C = 0$ and $B = 0$, i.e.

(17) $$a_0 + a_1 z + \cdots + a_m z^m = 0,$$

(18) $$(a_0 b_1) + \sum_{i=2}^{m} i (a_0 b_i) z^{i-1} + \sum_{\substack{j,i=1\\ i>j}}^{m} (i-j) (a_j b_i) z^{i+j-1} = 0,$$

(19) $$|(a_0 b_1)| - \sum_{i=2}^{m} i \, |(a_0 b_i)| \, r^{i-1} - \sum_{\substack{j,i=1\\ i>j}}^{m} (i-j) \, |(a_j b_i)| \, r^{i+j-1} = 0$$

similarly as in Section **5** of this paper serve as a basis for determining a radius of univalence, or even the maximal radius of univalence of rational functional f defined by (15).

In particular, the equations (18) and (19) obtain the following forms:

for $m=5$

(20) $$(a_0 b_1)+2(a_0 b_2)z+(3(a_0 b_3)+(a_1 b_2))z^2+(4(a_0 b_4)+2(a_1 b_3))z^3$$
$$+(5(a_0 b_5)+3(a_1 b_4)+(a_2 b_3))z^4+(4(a_1 b_5)+2(a_2 b_4))z^5$$
$$+(3(a_2 b_5)+(a_3 b_4))z^6+2(a_3 b_5)z^7+(a_4 b_5)z^8=0,$$

(21) $$|(a_0 b_1)|-2|(a_0 b_2)|r-(3|(a_0 b_3)|+|(a_1 b_2)|)r^2-(4|(a_0 b_4)|+2|(a_1 b_3)|)r^3$$
$$-(5|(a_0 b_5)|+3|(a_1 b_4)|+|(a_2 b_3)|)r^4-(4|(a_1 b_5)|+2|(a_2 b_4)|)r^5$$
$$-(3|(a_2 b_5)|+|(a_3 b_4)|)r^6-2|(a_3 b_5)|r^7-|(a_4 b_5)|r^8=0;$$

for $m=7$

(22) $$(a_0 b_1)+2(a_0 b_2)z+(3(a_0 b_3)+(a_1 b_2))z^2+(4(a_0 b_4)+2(a_1 b_3))z^3$$
$$+(5(a_0 b_5)+3(a_1 b_4)+(a_2 b_3))z^4+(6(a_0 b_6)+4(a_1 b_5)+2(a_2 b_4))z^5$$
$$+(7(a_0 b_7)+5(a_1 b_6)+3(a_2 b_5)+(a_3 b_4))z^6$$
$$+(6(a_1 b_7)+4(a_2 b_6)+2(a_3 b_5))z^7+(5(a_2 b_7)+3(a_3 b_6)+(a_4 b_5))z^8$$
$$+(4(a_3 b_7)+2(a_4 b_6))z^9+(3(a_4 b_7)+(a_5 b_6))z^{10}+2(a_5 b_7)z^{11}+(a_6 b_7)z^{12}=0,$$

(23) $$|(a_0 b_1)|-2|(a_0 b_2)|r-(3|(a_0 b_3)|+|(a_1 b_2)|)r^2-(4|(a_0 b_4)|+2|(a_1 b_3)|)r^3$$
$$-(5|(a_0 b_5)|+3|(a_1 b_4)|+|(a_2 b_3)|)r^4-(6|(a_0 b_6)|+4|(a_1 b_5)|+2|(a_2 b_4)|)r^5$$
$$-(7|(a_0 b_7)|+5|(a_1 b_6)|+3|(a_2 b_5)|+|(a_3 b_4)|)r^6$$
$$-(6|(a_1 b_7)|+4|(a_2 b_6)|+2|(a_3 b_5)|)r^7+(5|(a_2 b_7)|+3|(a_3 b_6)|+|(a_4 b_5)|)r^8$$
$$-(4|(a_3 b_7)|+2(a_4 b_6)|)r^9-(3|(a_4 b_7)|+|(a_5 b_6)|)r^{10}$$
$$-2|(a_5 b_7)|r^{11}-|(a_6 b_7)|r^{12}=0.$$

The formulas clearly possess a high degree of symmetry.

Formulas (22) and (23), for $a_6=a_7=b_6=b_7=0$, reduce to (20) and (21), respectively.

7. We consulted a large number of papers and, in particular, monographs [2] — [6], as well as the thorough exposition [7] and we did not find the results given here, though they are elementary. The Theorem 1 given here in Section **1** seems to be particularly interesting.

Prof. D. Mitrović and Prof. D. D. Adamović read this paper in manuscript and made valuable comments.

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