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667. ANALOGIES BETWEEN DIFFERENTIAL AND DIFFERENCE EQUATIONS, II: FIRST AND NONPARABOLIC SECOND ORDER PARTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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1. Introduction. The operators $\Delta_x, \Delta_y, \Delta_{xx}, \Delta_{xy}, \Delta_{yy}$ are defined as follows:

$$\Delta_x u(x, y) = u(x+1, y) - u(x, y); \quad \Delta_y u(x, y) = u(x, y+1) - u(x, y);$$

$$\Delta_{xx} u = \Delta_x(\Delta_x u); \quad \Delta_{xy} u = \Delta_y(\Delta_x u); \quad \Delta_{yy} u = \Delta_y(\Delta_y u).$$

The object of this note is to examine the partial difference equations

$$A\,\Delta_x u + B\,\Delta_y u + Cu = 0,$$

$$A\,\Delta_{xx} u + 2\,B\,\Delta_{xy} u + C\,\Delta_{yy} u + 2\,D\,\Delta_x u + 2\,E\,\Delta_y u + Fu = 0$$

together with the analogous partial differential equations

$$Au_x + Bu_y + Cu = 0,$$

$$Au_{xx} + 2\,Bu_{xy} + Cu_{yy} + 2\,Du_x + 2\,Eu_y + Fu = 0,$$

where $A \neq 0$, B, C, D, E, F are constants, and, in particular, to examine whether their respective solutions are analogous to each other.

2. First order equations. We only give the final results.

The general solution of the equation

(2.1) $$Au_x + Bu_y + Cu = 0$$

is

$$u = e^{-(C/A)x} f(Bx - Ay), \quad A \neq 0; \quad u = e^{-(C/B)y} f(x), \quad A = 0, \quad B \neq 0.$$

The general solution of the equation

(2.2) $$A\,\Delta_x u + B\,\Delta_y u + Cu = 0$$

is

$$u = \left(\frac{A+B-C}{A}\right)^x \sum_{\nu=0}^{x} \binom{x}{\nu} \left(\frac{B}{C-A-B}\right)^\nu f(y+\nu) \qquad (A+B \neq C, \quad A \neq 0)$$

(2.3) $$u = \left(\frac{B-C}{B}\right)^y f(x) \qquad (A+B \neq C; \quad A = 0; \quad B \neq 0)$$

(2.4) $$u = \left(-\frac{B}{A}\right)^x f(x+y) \qquad (A+B = C; \quad AB \neq 0),$$

where in all cases f is an arbitrary function.

It is evident that the general solution of (2.2) is not analogous to the general solution of (2.1), though both equations can always be solved. In particular, if $A+B\neq C$, $A\neq 0$, the general solution of (2.2) cannot be expressed by means of a formula which contains only one appearance of the arbitrary function f.

It will be convenient to introduce the following notion. If the general solution of (2.2) has the form (2.3) or (2.4), we shall say that it has a *simple general solution*. In other words, the equation (2.2) does not have a simple general solution only in the case when $A+B\neq C$, $A\neq 0$.

3. Second order nonparabolic partial differential equations. There are two well-known general methods for solving the partial differential equation

$$Au_{xx}+\ +2\,Bu_{xy}+Cu_{yy}+2\,Du_x+2\,Eu_y+Fu=0, \tag{3.1}$$

under the conditions

$$\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}=0,\quad B^2-AC\neq 0,\quad A\neq 0. \tag{3.2}$$

One method is to replace the equation (3.1) by two first order equations

$$u_x+M_1u_y+N_1u=0;\quad u_x+M_2u_y+N_2u=0, \tag{3.3}$$

where (M_1, N_1) and (M_2, N_2) are obtained from the system

$$K+AM=2\,B,\quad KM=C,\quad L+AN=2\,D,\quad ML+KN=2\,E.$$

Notice that the second condition (3.2) implies that $(M_1, N_1)\neq(M_2, N_2)$, and hence the two first order equations (3.3) cannot coincide. If $u=F_1(x, y)$ and $u=F_2(x, y)$ are the respective general solutions of the equations (3.3) then $u=F_1(x, y)+F_2(x, y)$ is the general solution of the equation (3.1).

The second method is to put $u=\exp(ax+by)$ into (3.1). The resulting quadratic equation in a, b yields $a=Pb+Q\pm\sqrt{Rb^2+Sb+T}$, where P, Q, R, S, T depend on A, B, C, D, E, F. The first condition (3.2) implies that $S^2=4\,RT$, which means that $Rb^2+Sb+T=(Gb+H)^2$ and hence $a=Pb+Q\pm(Gb+H)$. Then the functions $C_k\exp\big((P+G)\,kx+ky+(Q+H)\,x\big)$ and $D_k\exp\big((P-G)\,kx+ky+(Q-H)\,x\big)$, where C_k, $D_k\,(k=1, 2, \ldots)$ are arbitrary constants, are solutions of (3.1) and hence so is

$$u=\sum_{k=1}^{+\infty}C_k\exp\big((P+G)kx+ky+(Q+H)x\big)+\sum_{k=1}^{+\infty}D_k\exp\big((P-G)kx+ky+(Q-H)x\big).$$

Thus we obtain the general solution of (3.1) in the form

$$u=e^{(Q+H)x}f\big((P+G)\,x+y\big)+e^{(Q-H)x}g\big((P-G)\,x+y\big)$$

where

$$f(t)=:\sum_{k=1}^{+\infty}C_k\,e^{kt},\qquad g(t)=:\sum_{k=1}^{+\infty}D_k\,e^{kt}$$

are arbitrary functions.

The two methods are equivalent, as shown in [**1**].

4. Second order nonparabolic partial difference equations — reduction to two first order equations. The partial difference equation

$$(4.1) \qquad A\,\Delta_{xx}u+2\,B\,\Delta_{xy}u+C\,\Delta_{yy}u+2\,D\,\Delta_x u+2\,E\,\Delta_y u+Fu=0$$

subjected to the conditions (3.2) can be replaced by two first order equations

$$(4.2) \qquad \Delta_x u+M_1\,\Delta_y u+N_1 u=0; \quad \Delta_x u+M_2\,\Delta_y u+N_2 u=0,$$

where M_1, M_2, N_1, N_2 are defined as before. Moreover, if $u=F_1(x, y)$ and $u=F_2(x, y)$ are respective general solutions of the equations (4.2), then the general solution of (4.1) is $u=F_1(x, y)+F_2(x, y)$.

We therefore see that there is a complete analogy between equations (3.1) and (4.1), when they are solved by reduction to first order equations.

5. Second order nonparabolic partial difference equations — method of infinite series. Put $u=a^x b^y$ (a, b constants) into (4.1). We arrive at a quadratic equation in a and b, which, after solving for a, yields

$$(5.1) \qquad a=\frac{1}{A}\Big(-Bb+(A+B-D)\pm$$

$$\pm\sqrt{(B^2-AC)\,b^2-2\,(B^2-BD-AC+AE)\,b+(B^2+D^2-2\,BD-AC+2\,AE-AF)}\Big).$$

From the first condition (3.2) we get $F=(2\,BDE-CD^2+AE^2)/(B^2-AC)$ and substituting this value of F into (5.1) we find

$$a=\frac{1}{A}\Bigg(-Bb+(A+B-D)\pm$$

$$\pm\sqrt{(B^2-AC)\,b^2-2\,(B^2-BD-AC+AE)\,b+\frac{(B^2-BD-AC+AE)^2}{B^2-AC}}\Bigg)$$

i. e.

$$a=p_1 b+q_1 \quad \text{or} \quad a=p_2 b+q_2,$$

where

$$p_1=\frac{1}{A}\big(-B+\sqrt{B^2-AC}\big), \qquad q_1=\frac{1}{A}\big(A+B-D-(B^2-BD-AC+AE)/\sqrt{B^2-AC}\big),$$

$$p_2=\frac{1}{A}\big(-B-\sqrt{B^2-AC}\big), \qquad q_2=\frac{1}{A}\big(A+B-D+(B^2-BD-AC+AE)/\sqrt{B^2-AC}\big).$$

We therefore obtain the following solutions of the equation (4.1):

$$u=(p_1\,b+q_1)^x\,b^y \quad \text{and} \quad u=(p_2\,b+q_2)^x\,b^y \qquad (b \text{ arbitrary}),$$

from which we can form the following solution

$$(5.2) \qquad u=\sum_{k=1}^{+\infty} C_k\,(p_1\,k+q_1)^x\,k^y+\sum_{k=1}^{+\infty} D_k\,(p_2\,k+q_2)^x\,k^y,$$

where C_k, D_k are arbitrary constants.

The series which appear in (5.2) can be expressed in closed form, as in the case of partial differential equations, only if one of the following conditions is fulfilled:

$$(5.3)\qquad \begin{array}{llll} \text{(i)} & p_1=0,\quad p_2=0; & \text{(ii)} & p_1=0,\quad q_2=0; \\ \text{(iii)} & q_1=0,\quad p_2=0; & \text{(iv)} & q_1=0,\quad q_2=0. \end{array}$$

Indeed, in those cases we get:

$$\text{(i)}\quad u=\sum_{k=1}^{+\infty} C_k q_1{}^x k^y+\sum_{k=1}^{+\infty} D_k q_2{}^x k^y=q_1{}^x f(y)+q_2{}^x g(y);$$

$$\text{(ii)}\quad u=\sum_{k=1}^{+\infty} C_k q_1{}^x k^y+\sum_{k=1}^{+\infty} D_k p_2{}^x k^x k^y=q_1{}^x f(y)+p_2{}^x g(x+y);$$

$$\text{(iii)}\quad u=\sum_{k=1}^{+\infty} C_k p_1{}^x k^x k^y+\sum_{k=1}^{+\infty} D_k q_2{}^x k^y=p_1{}^x f(x+y)+q_2{}^x g(y);$$

$$\text{(iv)}\quad u=\sum_{k=1}^{+\infty} C_k p_1{}^x k^x k^y+\sum_{k=1}^{+\infty} D_k p_2{}^x k^x k^y=p_1{}^x f(x+y)+p_2{}^x g(x+y),$$

where $f(t)=:\sum_{k=1}^{+\infty} C_k k^t$, $g(t)=:\sum_{k=1}^{+\infty} D_k k^t$ are arbitrary functions, and those are the general solutions of the equation (4.1).

Hence, it would seem that the analogy between partial differential and partial difference equations breaks down. Namely, the general solution of equation (3.1) can be obtained in closed form by means of infinite series provided that the conditions (3.2) are fulfilled, whereas the general solution of the corresponding equation (4.1) can be obtained in closed form by means of infinite series if, in addition to (3.2), we also have one of the conditions (5.3).

6. A comparison of the two methods. It will be shown here that the reason in the apparent disagreement between the two exposed methods lies in the fact that the first order equations (2.1) and (2.2) do not have analogous forms of their general solutions.

Namely, if the first order equations (4.2) are requested to have simple general solutions, then according to the results of section 2, we conclude that one of the following conditions must hold:

$$(6.1)\qquad \begin{array}{llll} \text{(i)} & M_1=0,\quad M_2=0; & \text{(ii)} & M_1=0,\quad 1+M_2=N_2; \\ \text{(iii)} & 1+M_1=N_1,\quad M_2=0; & \text{(iv)} & 1+M_1=N_1,\quad 1+M_2=N_2. \end{array}$$

A straightforward calculation shows that the conditions (6.1) are equivalent to the conditions (5.3). In other words, the infinite series which appear in (5.2) can be expressed in closed form (in an analogous manner as in the case of partial differential equations) if and only if the associated first order equations (4.2) have simple general solutions.

Suppose now that none of the conditions (5.3) is fulfilled, i. e. that the equations (4.2) do not have simple general solutions. Then, starting from (5.2) we have, formally,

$$u=\sum_{k=0}^{+\infty} C_k\left(\sum_{\nu=0}^{x}\binom{x}{\nu}p_1{}^{\nu}k^{\nu}q_1{}^{x-\nu}\right)k^y+\sum_{k=0}^{+\infty} D_k\left(\sum_{\nu=0}^{x}\binom{x}{\nu}p_2{}^{\nu}k^{\nu}q_2{}^{x-\nu}\right)k^y$$

$$=q_1{}^x\sum_{\nu=0}^{x}\binom{x}{\nu}\left(\frac{p_1}{q_1}\right)^{\nu}\sum_{k=0}^{+\infty}C_k k^{y+\nu}+q_2{}^x\sum_{\nu=0}^{x}\binom{x}{\nu}\left(\frac{p_2}{q_2}\right)^{\nu}\sum_{k=0}^{+\infty}D_k k^{y+\nu}$$

$$=q_1{}^x\sum_{\nu=0}^{x}\binom{x}{\nu}\left(\frac{p_1}{q_1}\right)^{\nu}f(y+\nu)+q_2{}^x\sum_{\nu=0}^{x}\binom{x}{\nu}\left(\frac{p_2}{q_2}\right)^{\nu}g(y+\nu),$$

where f and g are arbitrary functions, defined as above.

It is not difficult to verify that the same solution is obtained by the method of section 4.

This shows that the two methods (the method of reduction to first order equations and the method of infinite series) are completely analogous for partial differential and partial difference equations. The forms of the solution of those equations are not analogous, owing to the fact that the general solutions of first order partial differential and partial difference equations are not analogous in form.

REFERENCE

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