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666. ON CONVEXITY PRESERVING MATRIX TRANSFORMATIONS*

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1. A sequence $a=(a_n)_{n=0}^{\infty}$ is said to be convex of order r if $\triangle^r a_n \geqq 0$, $n=0, 1, \ldots,$ where

$$\Delta^r a_n = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} a_{n+k}.$$

If $\|p_{n,i}\|$, $i=0, 1, \ldots, n$; $n=0, 1, \ldots,$ is a triangular matrix of real numbers, let $A(a)=(A_n(a))_{n=0}^{\infty}$ be the sequence defined as

$$A_n(a) = \sum_{k=0}^{n} p_{n,n-k}\, a_k, \qquad n=0, 1, \ldots . \tag{1}$$

If K_r denotes the set of all sequences which are convex of order r, then we say that the matrix transformation $A: a \rightarrow A(a)$ *preserves the r-convexity* if and only if for every $a \in K_r$ one has $A(a) \in K_r$: in other words, iff $A(K_r) \subseteq K_r$.

2. Let us present some particular sequences: if j is a non-negative integer we denote by $e_* = e_*(j) = (e_n(j))_{n=0}^{\infty}$ the sequence whose terms are

$$e_k(j) = \begin{cases} 0, & k=0, 1, \ldots, j-1 \\ C \cdot \binom{k}{j}, & k=j, j+1, j+2, \ldots \end{cases} \qquad C \in \mathbf{R} \backslash \{0\}.$$

Further, for a fixed integer s, $s \geqq 0$, let $f_* = f_*(s) = (f_n(s))_{n=0}^{\infty}$ defined as

$$f_k(s) = \begin{cases} 0, & k=0, 1, \ldots, s+r-1 \\ \binom{k-s-1}{r-1}, & k=s+r, s+r+1, \ldots . \end{cases}$$

Using the identity (see for instance [5], page 147)

$$\sum_{i=0}^{N} (-1)^{N-i} \binom{N}{i} \binom{a+mi}{k} = 0 \quad \text{if} \quad N > k,$$

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as well as the notation

$$s_0(n, k) = p_{n,k},$$

$$s_{m+1}(n, k) = \sum_{j=0}^{k} s_m(n, j) = \sum_{j=0}^{k} \binom{k+m-j}{m} p_{n,j} \qquad m = 0, 1, \dots,$$

$$X_n(m, k) = \begin{cases} 0, & n < k \\ s_m(n, n-k), & n \geq k \end{cases}, \tag{2}$$

the following proposition may be established:

Lemma 1. *If $e_*(j), f_*(s)$ are the sequences defined as above, then*

$$A_n(e_*(j)) = C \cdot X_n(j+1, j); \qquad A_n(f_*(s)) = X_n(r, s+r), \tag{3}$$

$$\Delta^i e_0(j) = \begin{cases} 0, & i \neq j \\ C, & i = j \end{cases}; \qquad \Delta^m f_0(s) = 0 \quad \text{if} \quad m \leq r-1, \tag{4}$$

$$\Delta^r e_i(j) = 0 \quad \text{for} \quad i = 0, 1, \dots \ (0 \leq j \leq r-1); \qquad \Delta^r f_i(s) = \begin{cases} 0, & i \neq s \\ 1, & i = s \end{cases}. \tag{5}$$

3. Professor D. S. Mitrinović (see [2]) has raised the problem of finding the set of all matrices $\|p_{n,i}\|$ which furnish matrix transformations $A : (a_n) \to (A_n(a))$ which preserve the r-convexity. Some kinds of such transformations were investigated in [1]–[3] and [6].

Our purpose is to present a solution for the case when A is defined by means of (1).

Lemma 2. *Let $a = (a_n)_{n=0}^{\infty}$ be an arbitrary sequence of real numbers. If $(A_n(a))_{n=0}^{\infty}$ is defined as in* (1), *then*

$$\Delta^r A_n(a) = \sum_{j=0}^{r-1} \Delta^j a_0 \cdot \Delta^r X_n(j+1, j) + \sum_{j=0}^{n} \Delta^r a_j \cdot \Delta^r X_n(r, j+r), \tag{6}$$

where $(X_n(m, k))_{n=0}^{\infty}$ is given in (2).

Proof. Let $K_0, \dots, K_{r-1}, C_0, \dots, C_n$ be such that

$$\Delta^r A_n(a) = \sum_{i=0}^{r-1} \Delta^i a_0 \cdot K_i + \sum_{i=0}^{n} C_i \cdot \Delta^r a_i. \tag{7}$$

Taking into account that $\Delta^r A_n(\cdot)$ is a linear map we observe that the numbers K_i and C_i does not depend on the sequence $(a_n)_{n=0}^{\infty}$. Moreover, if we identify in (7) the coefficients of $a_0, a_1, \dots, a_{n+r}$ one finds a linear system of $n+r+1$ equations with the unknowns $K_0, \dots, K_{r-1}, C_0, \dots, C_n$. It is observed that the determinant of this system is $\neq 0$: for a particular case see [4]. Setting in (7) $a = e_*(j)$, $0 \leq j \leq r-1$, one finds (see (3)–(5))

$$K_j = \frac{1}{C} \Delta^r A_n(e_*(j)) = \Delta^r X_n(j+1, j), \quad j = 0, 1, \dots, r-1.$$

If we select $a = f_*(s)$ then (7) together with (3)–(5) furnish us

$$C_s = \Delta^r A_n(f_*(s)) = \Delta^r X_n(r, s+r), \quad s = 0, 1, \dots$$

which proves (6).

Theorem 3. *Let $A_n(a)$ be defined as in* (1). *The matrix transformation $A: a \to (A_n(a))_{n=0}^{\infty}$ preserves the r-convexity if and only if*

(8) $\Delta^r X_n(j+1, j) = 0$ *for* $j = 0, 1, \ldots, r-1;$ $n = 0, 1, \ldots,$

and

(9) $\Delta^r X_n(r, i+r) \geqq 0$ *for* $i = 0, 1, \ldots, n;$ $n = 0, 1, \ldots.$

Proof. Let $a \in K_r$ and suppose that (8)–(9) are valid. The identity (6) implies

$$\Delta^r A_n(a) = \sum_{i=0}^{n} \Delta^r a_i \cdot \Delta^r X_n(r, i+r) \geqq 0, \quad n = 0, 1, \ldots.$$

Now suppose that $A(K_r) \subseteq K_r$. An element from K_r is the sequence $e_*(j)$ $(0 \leqq j \leqq r-1)$: indeed $\Delta^r e_n(j) = 0$, $n = 0, 1, \ldots$. Therefore we must have $\Delta^r A_n(e_*(j)) \geqq 0$. But this is equivalent with $C \cdot \Delta^r X_n(j+1, j) \geqq 0$ for every $C \in \mathbf{R} \setminus \{0\}$. In conclusion $\Delta^r X_n(j+1, j) = 0$, $j = 0, 1, \ldots, r-1$. Another r-convex sequence is $f_*(s)$. Using the fact that $\Delta^r A_n(f_*(s)) = \Delta^r X_n(r, s+r)$ the proof is complete.

Finally, we note that if $A_n^*(a) = \frac{1}{n+1} \sum_{k=0}^{n} a_k$ then the identity (6) furnishes us

$$\Delta^r A_n^*(a) = \sum_{j=0}^{n} \frac{(j+1)\,(j+2)\cdots(j+r)}{(n+1)\,(n+2)\cdots(n+r+1)} \Delta^r a_j. \tag{10}$$

The equality (10) implies that the implication $\Delta^r a_n \geqq 0 \Rightarrow \Delta^r A_n^*(a) \geqq 0$ is true.

REMARK OF THE EDITORIAL COMMITTEE

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REFERENCES

1. I. B. Lacković, S. K. Simić: *On weighted arithmetic means which are invariant with respect to k-th order convexity.* These Publications № **461** — № **497** (1974), 159—166.
2. D. S. Mitrinović, I. B. Lacković, M. S. Stanković: *Addenda to the monograph „Analytic inequalities", Part* II: *On some convex sequences connected with N. Ozeki's results.* Same Publications pp. 3—24.
3. N. Ozeki: *On the convex sequences* (IV). J. College Arts Sci. Chiba Univ. B **4** (1971), 1—4.
4. T. Popoviciu: *Introduction à la théorie des différences divisées.* Bull. Math. de la Soc. Roumaine des Sci., **42** (1940), 65—78.
5. I. J. Schwatt: *An introduction to the operations with series.* (Second edition), New York 1924.
6. P. M. Vasić, J. D. Keckić, I. B. Lacković, Ž. M. Mitrović: *Some properties of arithmetic means of real sequences.* Mat. Vesnik **9** (24) (1972), 205—212.

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