UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 634 — № 677 (1979), 62—69.

643. ON A GENERALIZATION OF CERTAIN RESULTS OF A. OSTROWSKI AND A. LUPAŞ

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0. In [1] G. GRÜSS has proved the following result (see also [2], [3]):

Let f and g be integrable functions on $[a, b]$. Then

$$|D(f, g)| \leq \frac{1}{4} (\operatorname*{Osc}_{[a,b]} f)(\operatorname*{Osc}_{[a,b]} g), \tag{0.1}$$

where

$$D(f, g) = \frac{1}{b-a} \int_a^b f(x) g(x) \, \mathrm{d}x - \frac{1}{(b-a)^2} \int_a^b f(x) \, \mathrm{d}x \int_a^b g(x) \, \mathrm{d}x.$$

The constant $\frac{1}{4}$ is the best possible one.

A. OSTROWSKI ([3]) has proved a certain class of inequalities connected with (0.1) imposing stronger conditions for f and g.

For instance, if g is bounded and measurable on $[a, b]$ and $f' \in L^2(a, b)$, then

$$|D(f, g)| \leq \frac{b-a}{4\sqrt{2}} (\operatorname*{Osc}_{[a,b]} g) \sqrt{\frac{1}{b-a} \int_a^b f'(x)^2 \, \mathrm{d}x} \tag{0.2}$$

holds.

Besides, if $g' \in L^2(a, b)$, then

$$|D(f, g)| \leq \frac{(b-a)^2}{8} \sqrt{\left(\frac{1}{b-a} \int_a^b f'(x)^2 \mathrm{d}x\right)\left(\frac{1}{b-a} \int_a^b g'(x)^2 \, \mathrm{d}x\right)}. \tag{0.3}$$

A. LUPAŞ ([4]) found the best possible constant in the last inequality: $\frac{(b-a)^2}{\pi^2}$. Also, A. LUPAŞ in [4] obtained the following inequality

$$|D(f, g)| \leq \frac{b-a}{2\pi} (\operatorname*{Osc}_{[a,b]} g) \sqrt{\frac{1}{b-a} \int_a^b f'(x)^2 \, \mathrm{d}x}, \tag{0.4}$$

which is stronger than the inequality (0.2).

We shall give some generalizations of inequalities (0.3) and (0.1) in our paper. Namely, instead of bounds for $D(f, g)$ we shall find bounds for $T(f, g) = A(fg; p) - A(f; p)\, A(g; p)$, where

$$A(f; p) = \frac{\int_a^b p(x) f(x)\, dx}{\int_a^b p(x)\, dx}.$$

$A(f; p)$ is a special case of OSTROWSKI's general means $M(f)$ introduced in [3].

1. Let $W_r^2[a, b]$ be the space of all functions u which are locally absolutely continuous on (a, b), with $\int_a^b r u'^2\, dx < +\infty$.

On account of the Theorem 3.1 of BEESACK ([5]) we can prove the following auxilary result:

Lemma 1. *Let* $-\infty \leqq a < b \leqq +\infty$, *and let p be positive and continuous on* (a, b) *with* $\int_a^b p\, dt = P < +\infty$. *Set* $r(x) = 1/p(x)$. *Then, if u is an integral on* $[a, b)$ *with* $u(a) = 0$ *and* $\int_a^b r u'^2\, dx < +\infty$ *we have*

$$\int_a^b p(x)\, u(x)^2\, dx \leqq \frac{4P^2}{\pi^2} \int_a^b r(x)\, u'(x)^2\, dx \tag{1.1}$$

with equality if and only if u is given by

$$u(x) = B \sin\left(\frac{\pi}{2P} \int_a^x p\, dt\right) \tag{1.2}$$

where B is arbitrary constant.

Proof. The hypotheses on u implies that $u(x) = \int_a^x u'(t)\, dt$ for $a \leqq x < b$. If a is finite this is equivalent to saying that $u(a) = 0$ and u is locally absolutely continuous on $[a, b)$. To prove (1.1), set $u = yz$, where $y(x) = \sin\left(\sqrt{\lambda_0} \int_a^x p\, dt\right)$ for $x \in [a, b]$, with $\lambda_0 = \frac{\pi^2}{4P^2}$. It is easy to verify that $(ry')' = -\lambda_0 p y$ on (a, b).

Now if $a<\alpha<\beta<b$, we have

$$\int_\alpha^\beta ru'^2\,dx=\int_\alpha^\beta r(y^2z'^2+2zz'yy'+y'^2z^2)\,dx$$

$$=\int_\alpha^\beta ry^2z'^2\,dx+\int_\alpha^\beta ry'^2z^2\,dx+ryy'z^2\Big|_\alpha^\beta-\int_\alpha^\beta z^2[y(ry')'+ry'^2]\,dx$$

$$=\int_\alpha^\beta ry^2z'^2\,dx+r\left(\frac{y'}{y}\right)u^2\Big|_\alpha^\beta+\lambda_0\int_\alpha^\beta pu^2\,dx$$

$$\geqq r(y'/y)u^2\Big|_\alpha^\beta+\lambda_0\int_\alpha^\beta pu^2\,dx.$$

Hence

$$\int_\alpha^\beta pu^2\,dx\leqq\lambda_0^{-1}\int_\alpha^\beta ru'^2\,dx+\lambda_0^{-1}\left\{\sqrt{\lambda_0}\,u(x)^2\operatorname{ctg}\left(\sqrt{\lambda_0}\int_a^x p\,dt\right)\right\}\Big|_\beta^\alpha. \tag{1.3}$$

Since

$$0\leqq u(\alpha)^2=\left(\int_a^\alpha u'\,dt\right)^2\leqq\left(\int_a^\alpha\sqrt{p}\sqrt{r}\,|u'|\,dt\right)^2\leqq\left(\int_a^\alpha p\,dt\right)\left(\int_a^\alpha ru'^2\,dt\right),$$

i.e.,

$$0\leqq\frac{u(\alpha)^2}{\sin\left(\sqrt{\lambda_0}\int_a^\alpha p\,dt\right)}\leqq\left(\int_a^\alpha ru'^2\,dt\right)\frac{\int_a^\alpha p\,dt}{\sin\left(\sqrt{\lambda_0}\int_a^\alpha p\,dt\right)}\to 0\text{ as }\alpha\to a+,$$

we have

$$u(\alpha)^2\operatorname{ctg}\left(\sqrt{\lambda_0}\int_a^\alpha p\,dt\right)\to 0\text{ as }\alpha\to a+\cdot$$

Then, from (1.3) it follows

$$\int_a^\beta pu^2\,dx\leqq\lambda_0^{-1}\int_a^\beta ru'^2\,dx-\lambda_0^{-1/2}u(\beta)^2\operatorname{ctg}\left(\sqrt{\lambda_0}\int_a^\beta p\,dt\right)\leqq\lambda_0^{-1}\int_a^\beta ru'^2\,dx,$$

where $a<\beta<b$. Now let $\beta\to b-$ to obtain the inequality (1.1).

The above proof shows that equality can hold in (1.1) only if $z'=0$, or $u=By$ for some constant B. Moreover, for any such u, we do have $u(x)=\int_a^x u'\,dt$, and $u\in W_r^2[a,b]$ as one easily verifies, so that u is an admissible function. By direct substitution one sees that equality does hold in (1.1) for such u.

The following result can be similarly proved (see Theorem 3.2 from [5]).

Lemma 2. *Let* $-\infty \leqq a < b \leqq +\infty$, *and let* p *be positive and continuous on* (a, b) *with* $\int_a^b p\mathrm{d}t = P < +\infty$. *Set* $r(x) = 1/p(x)$. *If* u *is an integral on* $(a, b]$ *with* $u(b) = 0$, *and* $\int_a^b ru'^2\,\mathrm{d}x < +\infty$, *the inequality* (1.1) *still holds, with equality if and only if* u *is given by*

$$u(x) = B_1 \sin\left(\frac{\pi}{2P}\int_x^b p\,\mathrm{d}t\right),$$

where B_1 *is arbitrary constant.*

Theorem 1. *Let* p *be positive and continuous on* (a, b) *with* $\int_a^b p\mathrm{d}t = P < +\infty$. *Set* $r(x) = 1/p(x)$, *and let* $a < \xi < b$. *Then for all* $F \in W_r^2[a, b]$ *the inequality*

$$(1.4)\qquad \int_a^b p(F(x) - F(\xi))^2\,\mathrm{d}x \leqq \frac{4}{\pi^2}\max\left\{\left(\int_a^\xi p\,\mathrm{d}t\right)^2, \left(\int_\xi^b p\,\mathrm{d}t\right)^2\right\}\int_a^b rF'(x)^2\,\mathrm{d}x$$

holds.

Equality in (1.4) *holds if and only if*

$$F(x) = B_2 + \begin{cases} B_1 \sin\left(\dfrac{\pi}{2}\dfrac{\int_a^x p\,\mathrm{d}t}{\int_a^\xi p\,\mathrm{d}t}\right) h(q) & (a \leqq x \leqq \xi), \\[2ex] B_1' \sin\left(\dfrac{\pi}{2}\dfrac{\int_x^b p\,\mathrm{d}t}{\int_\xi^b p\,\mathrm{d}t}\right) h(-q) & (\xi \leqq x \leqq b), \end{cases}$$

where B_1, B_1', B_2 *are arbitrary constants,* $q = \int_a^\xi p\,\mathrm{d}t - \int_\xi^b p\,\mathrm{d}t$ *and* h *is Heaviside's function.*

Proof. Let $a < \xi < b$. Applying Lemma 2 and Lemma 1 on the right hand side of equality

$$\int_a^b p(x)(F(x) - F(\xi))^2\,\mathrm{d}x = \int_a^\xi p(x)(F(x) - F(\xi))^2\,\mathrm{d}x + \int_\xi^b p(x)(F(x) - F(\xi))^2\,\mathrm{d}x,$$

we obtain (1.4). Notice that $x \mapsto u = F(x) - F(\xi)$ has the required behaviour at $x = \xi$.

Corollary 1. *Let functions* p *and* r *satisfy the conditions as in Theorem* 1 *and let* ξ *be such that*

$$(1.5)\qquad \int_a^\xi p\,\mathrm{d}t = \int_\xi^b p\,\mathrm{d}t.$$

Then

(1.6) $$\int_a^b p(x)\,(F(x)-F(\xi))^2\,dx \leq \left(\frac{P}{\pi}\right)^2 \int_a^b r(x)\,F'(x)^2\,dx,$$

with equality if and only if

$$F(x)=B_2+\begin{cases} B_1 \sin\left(\dfrac{\pi}{P}\displaystyle\int_x^\xi p\,dt\right) & (a\leq x\leq \xi),\\ B_1' \sin\left(\dfrac{\pi}{P}\displaystyle\int_\xi^x p\,dt\right) & (\xi\leq x\leq b),\end{cases}$$

where B_1, B_1', B_2 are arbitrary constants.

Proof. Since

$$Q=\max\left\{\int_a^\xi p\,dt,\ \int_\xi^b p\,dt\right\}=\frac{1}{2}\left\{\int_a^b p\,dt+\left|\int_\xi^b p\,dt-\int_a^\xi p\,dt\right|\right\},$$

with regard to (1.5), we have $Q=\frac{1}{2}P$.

Then, Corollary 1 follows from Theorem 1.

REMARK 1. Notice that (1.6) holds only for the single ξ such that (1.5) holds, and not for all ξ.

2. Define $\|h\|_r=\left(\int_a^b r(x)\,h(x)^2\,dx\right)^{1/2}$.

Theorem 2. *Let $p\in C(a,b)$, $p(x)>0$, $\int_a^b p\,dt=p<+\infty$ and $r(x)=1/p(x)$. If $f,g\in W_r^2[a,b]$, the inequality*

(2.1) $$|T(f,g)|\leq\frac{P}{\pi^2}\|f'\|_r\|g'\|_r$$

holds. If

(2.2) $$f(x)=A+B\sin\theta(x),\quad g(x)=C+D\sin\theta(x),$$

where $\theta(x)=\frac{\pi}{2P}\left(\int_x^b p\,dt-\int_a^x p\,dt\right)$, *the equality appears in* (2.1).

Proof. Let us prove at first the existence of the integrals $\int_a^b pf(x)^2\,dx$ and $\int_a^b pf(x)\,dx$.

Let $\xi \in (a, b)$. According to the Theorem 1 we have

$$\int_a^b p\,(f(x)-f(\xi))^2\,\mathrm{d}x < +\infty.$$

Hence also

$$\int_a^b p\,|f(x)-f(\xi)|\,\mathrm{d}x \le \Big(\int_a^b p\,\mathrm{d}x\Big)^{1/2}\Big(\int_a^b p\,(f(x)-f(\xi))^2\,\mathrm{d}x\Big)^{1/2} < +\infty,$$

so $\int_a^b p\,(f(x)-f(\xi))\,\mathrm{d}x$ exists, and since $\int_a^b p\,\mathrm{d}x$ exists, so does $\int_a^b pf(x)\,\mathrm{d}x$.

But then, since

$$\int_a^b p\,(f(x)-f(\xi))^2\,\mathrm{d}x = \int_a^b pf(x)^2\,\mathrm{d}x - 2f(\xi)\int_a^b pf(x)\,\mathrm{d}x + f(\xi)^2\int_a^b p\,\mathrm{d}x.$$

it also follows that $\int_a^b pf(x)^2\,\mathrm{d}x$ exists.

Since

$$\begin{aligned} T(f,f) &= A(f;p) - A(f;p)^2 \\ &= \frac{1}{P}\Big\{\int_a^b pf(x)^2\,\mathrm{d}x - A(f;p)\int_a^b pf(x)\,\mathrm{d}x\Big\} \\ &= \frac{1}{P}\int_a^b p\,\{f(x)^2 - f(x)\,A(f;p) - f(x)f(\xi) + f(\xi)\,A(f;p)\}\,\mathrm{d}x \\ &= \frac{1}{P}\int_a^b p\,(f(x)-A(f;p))\,(f(x)-f(\xi))\mathrm{d}x, \end{aligned}$$

and

$$T(f,f) = \frac{1}{P}\int_a^b p\,(f(x)-A(f;p))^2\,\mathrm{d}x \ge 0,$$

applying Cauchy's inequality, we obtain

$$\begin{aligned} |T(f,f)|^2 &\le \frac{1}{P^2}\int_a^b p\,(f(x)-A(f;p))^2\,\mathrm{d}x \int_a^b p\,(f(x)-f(\xi))^2\,\mathrm{d}x \\ &= \frac{1}{P}\,T(f,f)\int_a^b p\,(f(x)-f(\xi))^2\,\mathrm{d}x, \end{aligned}$$

i.e.,

$$0 \le T(f,f) \le \frac{1}{P}\int_a^b p\,(f(x)-f(\xi))^2\,\mathrm{d}x.$$

Chose ξ so that the condition (1.5) is satisfied. Then, with regard to (1.6), the last inequality becomes

$$T(f, f) \leq \frac{P}{\pi^2} \|f'\|_r^2.$$

Similarly,

$$T(g, g) \leq \frac{P}{\pi^2} \|g'\|_r^2.$$

Finally, using inequality $|T(f, g)|^2 \leqq T(f, f)\, T(g, g)$ proved in **[2]** for general means, including $A(f; p)$ as a special case, we obtain (2.1).

Inequality (2.1) can not be improved. Namely, when f and g, given with (2.2) are directly substituted in inequality (2.1), the equality is obtained.

REMARK 2. For $p(x) \equiv 1$, the inequality (2.1) reduces to inequality obtained by A. LUPAŞ ([4]).

On account of the inequality $|T(g, g)| \leqq \frac{1}{4} (\operatorname*{Osc}_{[a, b]} g)^2$ proved in **[2]** for general means, we can prove the following result:

Theorem 3. *Let* $p \in C(a, b)$, $p(x) > 0$, $\int_a^b p\, dt = P < +\infty$ *and* $r(x) = 1/p(x)$. *If* $f \in W_r^2[a, b]$ *and* g *is a measurable, bounded function, then*

$$|T(f, g)| \leq \frac{\sqrt{P}}{2\pi} \|f'\|_r (\operatorname*{Osc}_{[a, b]} g). \tag{2.3}$$

REMARK 3. For $p(x) = 1$, the inequality (2.3) reduces to LUPAŞ inequality ([4])

$$|D(f, g)| \leqq \frac{\sqrt{b-a}}{2\pi} \|f'\|_1 (\operatorname*{Osc}_{[a, b]} g).$$

REMARK 4. Using JACOBI's weight function $x \mapsto p(x) = (1-x)^\alpha (1+x)^\beta$, where $\alpha, \beta > -1$, $a = -1$, $b = 1$, the inequality (2.1) is reduced to

$$\left| \int_{-1}^{1} p(x) f(x) g(x)\, dx - \frac{1}{C} \int_{-1}^{1} p(x) f(x)\, dx \int_{-1}^{1} p(x) g(x)\, dx \right|$$

$$\leqq \frac{C^2}{\pi^2} \left(\int_{-1}^{1} \frac{f'(x)^2}{p(x)}\, dx \int_{-1}^{1} \frac{g'(x)^2}{p(x)}\, dx \right)^{1/2},$$

where $C = 2^{\alpha+\beta+1} \dfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}$.

3. The following theorem generalizes an A. LUPAŞ' inequality ([9]) in connection with an A. OSTROWSKI's result (see [10], [11], [12]).

Theorem 4. *Let* $p \in C(a, b)$, $p(x) > 0$, $\int_a^b p\, dt = P < +\infty$, $r(x) = 1/p(x)$ *and* $f \in W_r^2[a, b]$. *Then*

$$\left| f(x) - A(f; p) \right| \leqq \frac{2}{\pi \sqrt{P}} \max \left\{ \int_a^x p\, dt, \int_x^b p\, dt \right\} \|f'\|_r \tag{3.1}$$

for every $x \in (a, b)$.

Proof. Applying CAUCHY's inequality on the right hand side of equality

$$|f(x)-A(f;\,p)|^2=\frac{1}{P^2}\left|\int_a^b p\big(f(t)-f(x)\big)\,\mathrm{d}t\right|^2,$$

we obtain

$$|f(x)-A(f;\,p)|^2\leq\frac{1}{P}\int_a^b p\big(f(t)-f(x)\big)^2\,\mathrm{d}t,$$

which combined with (1.4) gives (3.1).

*

The authors are grateful to Prof. P. R. BEESACK for careful reading of the paper and useful suggestions which helped better and more complete formulations of the stated material.

Also, autors are grateful to Dr. A. LUPAŞ for useful suggestions in preparing the paper.

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