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640. ON THE JENSEN INEQUALITY

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In this paper we shall give some new results on the JENSEN inequality for convex functions and some applications on inequalities connecting the arithmetic and the geometric means.

1. The Jensen inequality

Theorem 1. *If $f\colon I\to \mathbf{R}$ is a convex function, $x=(x_1, \ldots, x_n)\in I^n$ $(n\geqq 2)$, p is positive n-tple, then*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)\leqq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \qquad \left(P_n=\sum_{i=1}^{n} p_i\right). \tag{1}$$

If f is a strictly convex function, inequality (1) *is strict unless $x_1=\cdots=x_n$.*

REMARKS. 1° Inequality (1) is the well-known JENSEN inequality. 2° On the history of the JENSEN inequality see [1].

From Theorem 1 we can get the inequality between the arithmetic and the geometric means:

Corollary 1. *Let x, p be two n-tples of positive numbers then*

$$G_n(x; p)\leqq A_n(x; p), \tag{2}$$

where

$$G_n(x; p)=\left(\prod_{i=1}^{n} x_i^{p_i}\right)^{1/P_n}, \qquad A_n(x; p)=\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i.$$

The equality occurs in (2) *if and only if $x_1=\cdots=x_n$.*

Corollary 2. *Let x and p be two n-tples of positive numbers and let function f satisfy* 1° *$f(x)$ is concave,* 2° *$xf(x)$ is convex for every $x\in[0, b]$. Then*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i)\leqq f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)\leqq \frac{\sum_{i=1}^{n} p_i x_i f(x_i)}{\sum_{i=1}^{n} p_i x_i}\leqq f\left(\frac{\sum_{i=1}^{n} p_i x_i^2}{\sum_{i=1}^{n} p_i x_i}\right). \tag{3}$$

For $f(x)=\log x$ we get the generalization of the result from [3], i.e.

$$G_n(x; p)\leqq A_n(x; p)\leqq G_n(x; px)\leqq A_n(x; px),$$

where $px=(p_1x_1, \ldots, p_nx_n)$.

2. The inverse Jensen inequality

Theorem 2. *Let p be a n-tple of real numbers such that*

(4) $$p_1>0, \quad p_i \leqq 0\,(i=2, \ldots, n), \quad P_n>0.$$

If $f: I \to \mathbf{R}$ is a convex function, $x \in I^n$, $\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \in I$, then the reverse of (1) *holds. The equality holds under the same conditions as in Theorem* 1.

Theorem 2 for $n=2$ was proved in [7].

Proof. By substitutions

(5) $$p_1=P_n,\; x_1=\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i; \quad p_i=-p_i, \quad x_i=x_i \qquad (i=2, \ldots, n)$$

from Theorem 1, we get Theorem 2.

Corollary 3. *Let x be a positive n-tple and let p be a real n-tple such that* (4) *holds. Then the reverse of* (2) *holds.*

Proof. Let $\sum_{i=1}^{n} p_i x_i>0$. For $f(x)=-\log x$, from Theorem 2 we get (2) with the reverse inequality. If $\sum_{i=1}^{n} p_i x_i<0$ this result is obvious.

Using (5), we can get Corollary 3 from Corollary 1, too.

3. Refinement of the Jensen inequality

Let I be a finite nonempty set of positive integers. If we define the index set function F by

$$F(I)=\sum_{i\in I} p_i f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)-\sum_{i\in I} p_i f(x_i),$$

and if

$$P_I=\sum_{i\in I} p_i, \quad A_I(x;p)=\frac{1}{P_I}\sum_{i\in I} p_i x_i, \quad G_I(x;p)=\left(\prod_{i\in I} x_i^{p_i}\right)^{1/P_I},$$

then the following theorem is valid:

Theorem 3. *If f: $[a, b]\to\mathbf{R}$ is a convex function on $[a, b]$, I and J finite nonempty sets of positive integers such that $I\cap J=\varnothing$, $p=(p_i)_{i\in I\cup J}$ and $x=(x_i)_{i\in I\cup J}$ are real sequences such that $x_i\in[a, b]$ $(i\in I\cup J)$, $A_M(x;p)\in[a, b]$ $(M=I, J, I\cup J)$, $P_{I\cup J}>0$.*

Let $P_I>0$ and $P_J>0$, then

(6) $$F(I\cup J)\leqq F(I)+F(J).$$

If $P_I P_J<0$ the sense of (6) *reverses. If f is a strictly convex function, equality in* (6) *holds if and only if $A_I(x;p)=A_J(x;p)$.*

Proof. By substitutions

$$x_1=A_I(x;p),\quad p_1=P_I,\quad x_2=A_J(x;p),\quad p_2=P_J,$$

from the JENSEN inequality for $n=2$, we get (6). Analogously from Theorem 2 for $n=2$, we get (6) with the reverse inequality.

Corollary 4. *If* $p_i\geqq 0$ $(i=1,\ldots,n)$, $I_k=\{1,\ldots,k\}$ *then*

$$F(I_n)\leqq F(I_{n-1})\leqq\cdots\leqq F(I_2)\leqq 0. \tag{7}$$

With the same assumption on p, x as in Theorem 2, we have the reverse inequalities in (7).

REMARKS. 3° Inequalities (6) and (7) for a positive sequence p was proved in [4].

4° Substituting I (i.e. $[a,b]$) by a convex set U from $\mathbf{R}^n$ and x_i by points from U, Theorems 1, 2, 3 remain valid even for convex functions in several variables with the same proofs as in single-dimensional case.

Now, let be

$$\rho(I)=P_I(A_I(x;p)-G_I(x;p))\quad(I\neq\varnothing),\quad \rho(\varnothing)=0;$$

$$\pi(I)=\left(\frac{A_I(x;p)}{G_I(x;p)}\right)^{P_I}\quad(I\neq\varnothing),\quad \pi(\varnothing)=0.$$

Then the following result is valid:

Corollary 5. *Let* $x=(x_i)_{i\in I\cup J}$ *be a positive sequence and let* $p=(p_i)_{i\in I\cup J}$ *be a real sequence such that* $P_{I\cup J}>0$.

(a) *Let* $P_I>0$, $P_J>0$. *Then*

$$\rho(I\cup J)\geqq\rho(I)+\rho(J). \tag{8}$$

If $P_IP_J<0$ *then the sense of* (8) *reverses. The equality holds in* (8) *if and only if* $G_I(x;p)=G_J(x;p)$.

(b) *Let* $A_M(x;p)>0$ $(M=I,J,I\cup J)$. *If* $P_I>0$, $P_J>0$, *then*

$$\pi(I\cup J)\geqq\pi(I)\,\pi(J), \tag{9}$$

and if $P_IP_J<0$ *then the sense of* (9) *reverses. The equality holds in* (9) *if and only if* $A_I(x;p)=A_J(x;p)$.

Proof. Let $P_I>0$, $P_J>0$. Inequalities (8) and (9) could be written in the following way

$$\frac{P_I}{P_{I\cup J}}G_I(x;p)+\frac{P_J}{P_{I\cup J}}G_J(x;p)\geqq G_I(x;p)^{P_I/P_{I\cup J}}\,G_J(x;p)^{P_J/P_{I\cup J}},$$

and

$$\frac{P_I}{P_{I\cup J}}A_I(x;p)+\frac{P_J}{P_{I\cup J}}A_J(x;p)\geqq A_I(x;p)^{P_I/P_{I\cup J}}\,A_J(x;p)^{P_J/P_{I\cup J}},$$

which is true on the basis of Corollary 1. Analogously using Corollary 3 we get the reverse inequalities.

REMARK. 5° Corollary 5 is the generalisation of the well-known EVERITT result (see, for example [1, p. 54]).

Corollary 6. *If p is a positive n-tple then*

(10) $$\rho(I_n) \geqq \rho(I_{n-1}) \geqq \cdots \geqq \rho(I_2) \geqq 0,$$

(11) $$\pi(I_n) \geqq \pi(I_{n-1}) \geqq \cdots \geqq \pi(I_2) \geqq 1,$$

(12) $$A_n(x; p) - G_n(x; p) \geqq \frac{1}{P_n} \max_{1 \leqq j,\, k \leqq n} \left(x_j p_j + x_k p_k - (p_j + p_k)\left(x_j^{p_j} x_k^{p_k}\right)^{\frac{1}{p_j + p_k}} \right),$$

(13) $$\frac{A_n(x; p)}{G_n(x; p)} \geqq \max_{1 \leqq j,\, k \leqq n} \left(\left(\frac{x_j p_j + x_k p_k}{p_j + p_k} \right)^{p_j + p_k} x_j^{-p_j} x_k^{-p_k} \right)^{1/P_n}.$$

If p is a real n-tple such that (4) holds and $\sum_{i=1}^{n} p_i x_i \geqq 0$, then the reverse inequalities in (10) and (11) and the following results hold:

(14) $$A_n(x; p) - G_n(x; p) \leqq \frac{1}{P_n} \min_{2 \leqq k \leqq n} \left(x_1 p_1 + x_k p_k - (p_1 + p_k)\left(x_1^{p_1} x_k^{p_k}\right)^{\frac{1}{p_1 + p_k}} \right),$$

(15) $$\frac{A_n(x; p)}{G_n(x; p)} \leqq \min_{2 \leqq k \leqq n} \left(\left(\frac{x_1 p_1 + x_k p_k}{p_1 + p_k} \right)^{p_1 + p_k} x_1^{-p_1} x_k^{-p_k} \right)^{1/P_n}.$$

REMARK. 6° The results from Corollary 6 are the generalization of the well-known RADO and POPOVICIU inequalities (see [1, pp. 49—51]).

4. Converses of the Jensen inequality

A converse of the JENSEN inequality is given in [5]:

Theorem 4. *If $f\colon I \to \mathbf{R}$ is a convex function, $x_i \in [m, M] \subseteq I$ $(i = 1, \ldots, n)$, p is a positive n-tple, then*

(16) $$\frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \leqq \frac{M - \bar{x}}{M - m} f(m) + \frac{\bar{x} - m}{M - m} f(M),$$

where $\bar{x} = A_n(\boldsymbol{x}; p)$.

The right-hand side is a nondecreasing function of M and a nonincreasing function of m. There is equality in (16) if and only if $x_i = m$ $(i \in J \subset \{1, \ldots, n\})$ and $x_i = M$ $(i \in \{1, \ldots, n\} \setminus J)$.

REMARK. 7° In [6] it was shown that Teorem 4 could be obtained from the JENSEN inequality.

In [2] the following two theorems were given:

Theorem 5. *If $f\colon I \to \mathbf{R}$, $f(x) > 0$, $f''(x) > 0$ for $x \in I$, $x_i \in I$ $(i = 1, \ldots, n)$, p is a positive n-tple then*

(17) $$\frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \leqq \lambda f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right),$$

where $m=\min(x)$, $M=\max(x)$ *and* λ *is a solution of the equation*

$$(18)\quad \lambda f\left(f'^{-1}\left(\frac{f(M)-f(m)}{\lambda(M-m)}\right)\right)=\frac{f(M)-f(m)}{M-m}f'^{-1}\left(\frac{f(M)-f(m)}{\lambda(M-m)}\right)+\frac{Mf(m)-mf(M)}{M-m}.$$

Theorem 6. *Let the conditions of the Theorem 5 be fulfilled. Then*

$$(19)\quad \frac{1}{P_n}\sum_{i=1}^{n}p_i f(x_i)\leqq \mu+f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i x_i\right),$$

where

$$\mu=\frac{f(M)-f(m)}{M-m}f'^{-1}\left(\frac{f(M)-f(m)}{M-m}\right)+\frac{Mf(m)-mf(M)}{M-m}-f\left(f'^{-1}\left(\frac{f(M)-f(m)}{M-m}\right)\right).$$

REMARKS. 8° The conditions for the validity of the equality in (17) are given in [**2**].

9° We note that inequality (16) is stronger than (17) and (19). Easily it is shown that inequality (16) could be used in the proof of (17) and (19) (see [**2**]), and then we get that λ and μ are non-decreasing functions of M and nonincreasing functions of m.

10° Analogously we can get

(a) If $f(m)=0$ and $f'(m)\neq 0$ then $\lambda=f(M)/(f'(m)(M-m))$;

(b) If $f(M)=0$ and $f'(M)\neq 0$ then $\lambda=-f(m)/(f'(M)(M-m))$.

11° For converse inequalities for inequality (2) see [**1**, pp. 63—64].

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