UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 634 — № 677 (1979), 36 — 41.

637. NOTES ON CONVEX FUNCTIONS VI: ON AN INEQUALITY FOR CONVEX FUNCTIONS PROVED BY A. LUPAŞ

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Using previously derived results, in our notes on convex functions, we will prove in the present paper that a pair of convex functions f and g satisfy the inequality (37) if and only if this inequality takes the form (38). This result is connected with earlier result (1) of A. Lupas.

1. In paper [1] A. LUPAŞ has proved the following statement:

Theorem 1. *If $x \mapsto f(x)$ and $x \mapsto g(x)$ are convex functions on $[a, b]$ then the following inequality*

$$(1) \qquad \int_a^b f(x) g(x)\,dx - \frac{1}{b-a}\Big(\int_a^b f(x)\,dx\Big)\Big(\int_a^b g(x)\,dx\Big)$$

$$\geqq \frac{12}{(b-a)^3}\Big(\int_a^b \Big(x-\frac{a+b}{2}\Big) f(x)\,dx\Big)\Big(\int_a^b \Big(x-\frac{a+b}{2}\Big) g(x)\,dx\Big)$$

holds where the equality is obtained if at least one of the functions f or g is linear on the segment $[a, b]$.

If among these suppositions we suppose that the following equality

$$(2) \qquad g\Big(\frac{a+b}{2}-x\Big)=g\Big(\frac{a+b}{2}+x\Big)$$

is satisfied for $x \in \Big[-\frac{b-a}{2}, \frac{b-a}{2}\Big]$ then (1) is reduced to the very known inequality

$$(3) \qquad \int_a^b f(x) g(x)\,dx \geqq \frac{1}{b-a}\Big(\int_a^b f(x)\,dx\Big)\Big(\int_a^b g(x)\,dx\Big).$$

If we introduce the substitution $x=a+(b-a)\,t$ in (1) we find that the inequality

$$(4) \qquad \int_0^1 F(t) G(t)\,dt - \Big(\int_0^1 F(t)\,dt\Big)\Big(\int_0^1 G(t)\,dt\Big)$$

$$\geqq 3\Big(\int_0^1 (1-2t) F(t)\,dt\Big)\Big(\int_0^1 (1-2t) G(t)\,dt\Big)$$

where the functions F and G are defined by

$$F(t)=f(a+(b-a)t),\quad G(t)=g(a+(b-a)t). \tag{5}$$

Besides, it is quiet clear, that the functions f and g are convex on $[a, b]$ if and only if the functions F and G, defined by (5) are convex on $[0, 1]$.

In connection with inequality (4) we will consider a bilinear operator A which is of the following form

$$A(f, g)=\int_0^1 f(t)g(t)\,dt-\Big(\int_0^1 f(t)\,dt\Big)\Big(\int_0^1 g(t)\,dt\Big) -K\Big(\int_0^1 p(t)f(t)\,dt\Big)\Big(\int_0^1 q(t)g(t)\,dt\Big) \tag{6}$$

where K is a real constant such that $K\neq 0$. Under the certain conditions we will show that for every pair of convex functions we have

$$A(f, g)\geqq 0 \tag{7}$$

if and only if the functions p and q are of the form which will be given below.

This result can be proved on the basis of our theorem which we have proved in paper [2]. This our theorem reads:

Theorem 2. *A continuous, bilinear operator A, defined on $C[a, b]\times C[a, b]$ satisfies the condition (7) for every pair of functions f and g, convex on $[0, 1]$, if and only if the following conditions are valid:*

$$A(e_i, e_j)=0 \qquad (i, j=0, 1), \tag{8}$$

$$A(e_i, w(t, c))=A(w(t, c), e_i)=0 \qquad (i=0, 1) \tag{9}$$

for every $c\in[0, 1]$, and

$$A(w(t, c_1), w(t, c_2))\geqq 0 \tag{10}$$

for every pair $c_1, c_2\in[0, 1]$, where the functions e_0, e_1 and w are defined by

$$e_0(t)=1,\quad e_1(t)=t,\quad w(t, c)=|t-c| \quad (t, c\in[0, 1]). \tag{11}$$

In virtue of (6) and theorem 2 we can directly obtain the following conditions:

$$A(e_0, e_0)=-K\Big(\int_0^1 p(t)\,dt\Big)\Big(\int_0^1 q(t)\,dt\Big)=0, \tag{12}$$

$$A(e_0, e_1)=-K\Big(\int_0^1 p(t)\,dt\Big)\Big(\int_0^1 tq(t)\,dt\Big)=0, \tag{13}$$

$$A(e_1, e_0)=-K\Big(\int_0^1 tp(t)\,dt\Big)\Big(\int_0^1 q(t)\,dt\Big)=0, \tag{14}$$

(15) $$A(e_1, e_1)=\frac{1}{12}-K\Big(\int_0^1 tp(t)\,dt\Big)\Big(\int_0^1 tq(t)\,dt\Big)=0,$$

(16) $$A(e_0, w(t,c))=-K\Big(\int_0^1 p(t)\,dt\Big)\Big(\int_0^1 |t-c|\,q(t)\,dt\Big)=0,$$

(17) $$A(w(t,c), e_0)=-K\Big(\int_0^1 |t-c|\,p(t)\,dt\Big)\Big(\int_0^1 q(t)\,dt\Big)=0,$$

(18) $$A(e_1, w(t,c))=\frac{c^3}{3}-\frac{c^2}{2}+\frac{1}{12}-K\Big(\int_0^1 tp(t)\,dt\Big)\Big(\int_0^1 |t-c|\,q(t)\,dt\Big)=0,$$

(19) $$A(w(t,c), e_1)=\frac{c^3}{3}-\frac{c^2}{2}+\frac{1}{12}-K\Big(\int_0^1 |t-c|\,p(t)\,dt\Big)\Big(\int_0^1 tq(t)\,dt\Big)=0,$$

(20) $$A(w(t,c_1), w(t,c_2))=\int_0^1 |t-c_1|\,|t-c_2|\,dt-\Big(\int_0^1 |t-c_1|\,dt\Big)\Big(\int_0^1 |t-c_2|\,dt\Big)$$
$$-K\Big(\int_0^1 |t-c_1|\,p(t)\,dt\Big)\Big(\int_0^1 |t-c_2|\,q(t)\,dt\Big)\geqq 0.$$

In such a way we obtain the following result:

Lemma 1. *The inequality*

(21) $$\int_0^1 f(t)g(t)\,dt-\Big(\int_0^1 f(t)\,dt\Big)\Big(\int_0^1 g(t)\,dt\Big)-K\Big(\int_0^1 p(t)f(t)\,dt\Big)\Big(\int_0^1 q(t)g(t)\,dt\Big)\geqq 0$$

is valid for every pair f and g of convex functions if and only if the conditions (12)—(20) *are satisfied.*

We will show further that under the certain conditions the function p and q can be determined in such a way that the conditions (12)—(20) are satisfied for these functions. For that purpose we will introduce the following denotations

(22) $$U(c)=\frac{c^3}{3}-\frac{c^2}{2}+\frac{1}{12},$$

(23) $$V(c_1, c_2)=A(w(t,c_1), w(t,c_2)),$$

(24) $$u=\int_0^1 p(t)\,dt,\quad v=\int_0^1 q(t)\,dt,$$

(25) $$P=\int_0^1 tp(t)\,dt,\quad Q=\int_0^1 tq(t)\,dt.$$

From the condition (15) it follows that we have

$$K \neq 0,\ P \neq 0,\ Q \neq 0, \tag{26}$$

where P and Q are given by (25). So, on the basis of (26) we have

$$K = \frac{1}{12PQ}. \tag{27}$$

By using (26) again on the basis of (13) and (14) we find that

$$u = v = 0 \tag{28}$$

where u and v are defined by (24). In such a way we find that if condition (28) is valid then the conditions (12), (16) and (17) are satisfied.

The conditions (26) and (18) i. e. (19) implies that

$$\int_0^1 |t-c|\, q(t)\, \mathrm{d}t = 12\, QU(c), \tag{29}$$

$$\int_0^1 |t-c|\, p(t)\, \mathrm{d}t = 12\, PU(c), \tag{30}$$

where the function $c \mapsto U(c)$ is defined by (22). By substitution of (29) and (30) in (20) we have

$$V(c_1, c_2) = \int_0^1 |t-c_1|\,|t-c_2|\, \mathrm{d}t - \Big(\int_0^1 |t-c_1|\, \mathrm{d}t\Big)\Big(\int_0^1 |t-c_2|\, \mathrm{d}t\Big) - 12\, U(c_1)\, U(c_2). \tag{31}$$

If we take $f(x) = |x-c_1|$ and $g(x) = |x-c_2|$ with $a=0$ and $b=1$ in (1) we obtain that $V(c_1, c_2) \geqq 0$ because in this case the difference of left and right side in (1) equals V, where V is of the form (31). In such a way, in virtue of the theorem 1 we conclude that the conditions (27), (29) and (30) are satisfied.

From what we have said about we can formulate the following lemma.

Lemma 2. *The conditions* (12)—(20) *are valid for a pair of functions p and q defined on* [0, 1] *if and only if the conditions* (26), (27), (28), (29) *and* (30) *are satisfied, where u, v, P, Q and V are defined by* (24), (25) *and* (22) *respectively.*

Suppose further that the functions p and q satisfy the conditions (29) and (30). Then, first of all it is clear that we must have

$$p(t) = mq(t) \qquad (0 \leqq t \leqq 1,\ m = \text{const}). \tag{32}$$

We will now consider the equation of the form

$$12\, U(c) \int_0^1 tp(t)\, \mathrm{d}t = \int_0^1 |t-c|\, p(t)\, \mathrm{d}t \tag{33}$$

where U is given by (22). We will also suppose that the function $t \mapsto p(t)$ is continuous on the segment [0, 1]. It can be directly verified that

$$\int_0^1 |t-c|\,p(t)\,dt = c\int_0^c p(t)\,dt - \int_0^c tp(t)\,dt + \int_c^1 tp(t)\,dt - c\int_c^1 p(t)\,dt. \tag{34}$$

By differentiation of the equality (33) with application of (34) we have that p must take the form

$$p(t) = 6(2t-1)P$$

where P is given by (25). Accordingly, the functions p and q must be of the following form

$$p(t) = k_1(2t-1), \quad q(t) = k_2(2t-1) \qquad (0 \leq t \leq 1). \tag{35}$$

So, in virtue of the above lemma 2 we find that the following lemma is valid.

Lemma 3. *Let us suppose that the functions p and q are continuous on the segment* [0, 1]. *Then the functions p and q satisfy the conditions* (12)—(20) *if and only if these functions are of the form* (35) *where the real constants are arbitrary chosen such that* $k_i \neq 0$ $(i = 1, 2)$ *and where we have*

$$K = \frac{3}{k_1 k_2}. \tag{36}$$

On the basis of the above given results it can be directly conclude that the following theorem is valid.

Theorem 3. *Suppose that the functions* $t \mapsto p(t)$ *and* $t \mapsto q(t)$ *are continuous on* [0, 1]. *Then the inequality of the form*

$$\int_0^1 f(x)g(x)\,dx - \Big(\int_0^1 f(x)\,dx\Big)\Big(\int_0^1 g(x)\,dx\Big) \geq K\Big(\int_0^1 p(x)f(x)\,dx\Big)\Big(\int_0^1 q(x)g(x)\,dx\Big) \tag{37}$$

holds for every pair of convex functions f and g if and only if these functions p and q are of the form (35) *where the real constants* $k_1 \neq 0$ *and* $k_2 \neq 0$ *are arbitrary and where the constant K is given by* (36). *In other words, for every pair of convex functions f and g the inequality* (37) *holds true if and only if the same inequality is of the form*

$$\int_0^1 f(x)g(x)\,dx - \Big(\int_0^1 f(x)\,dx\Big)\Big(\int_0^1 g(x)\,dx\Big) \geq 3\Big(\int_0^1 (2x-1)f(x)\,dx\Big)\Big(\int_0^1 (2x-1)g(x)\,dx\Big) \tag{38}$$

The inequality (38) reduces to the result of A. LUPAŞ (1) by the substitution $x=\frac{t-a}{b-a}$. The supposition of continuity of the above functions p and q can be weakened, but it is not of the essential importance for theorem 3.

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